MACMILLAN'S CANADIAN SCHOOL SERIES

SENIOR HIGH SCHOOL ALGEBRA

CRAWFORD

SUPERVISOR OF SCHOOLS
OFFICE

152
C89
S
1924

Digitized by the Internet Archive
in 2016

https://archive.org/details/seniorhighschool00craw

SUPERVISOR OF SCHOOLS
OFFICE

SENIOR
HIGH SCHOOL ALGEBRA

MACMILLAN'S CANADIAN SCHOOL SERIES

SENIOR HIGH SCHOOL ALGEBRA

BY

J. T. CRAWFORD, B.A.

CHIEF INSTRUCTOR IN MATHEMATICS, UNIVERSITY SCHOOLS,
ASSOCIATE PROFESSOR OF MATHEMATICS, COLLEGE OF EDUCATION,
UNIVERSITY OF TORONTO
AUTHOR OF "THE HIGH SCHOOL ALGEBRA"
JOINT AUTHOR OF "THE HIGH SCHOOL ARITHMETIC"

TORONTO
THE MACMILLAN COMPANY OF CANADA, LIMITED
1923

COPYRIGHT, CANADA, 1922
BY THE MACMILLAN COMPANY OF CANADA LIMITED
Reprinted 1922, 1923

PRINTED IN CANADA

PREFACE

This text is intended to cover all the work in Senior Algebra required for honour matriculation into any Canadian university.

From practical experience I have found that the best method of introducing Senior Algebra and of connecting it with the previous work of the pupil is by means of the solution of equations.

Teachers of mathematics will know, however, that many of the topics dealt with are not dependent upon each other, and therefore the order of the chapters may be changed at will to suit the individual teacher.

The text contains a large number of examples, but it is not intended that the pupil should work all of them at the first reading. It is advisable to reserve the more difficult parts of the theory and some of the more difficult examples for the second year's work.

The interest now taken by so many Canadians in bonds and annuities makes it imperative that these subjects should be given considerable space in a modern text.

A short chapter is added on Simple Probability and Insurance for those students who require this work for examination purposes.

The mathematical tables which are furnished will be found useful in solving problems on annuities and insurance.

The large number of miscellaneous examples at the end of the book will be valuable for review purposes. Many of them are sufficiently difficult to test the skill of those students who have special ability in mathematics.

CONTENTS

CHAPTER I

PAGE

Equations 1

Review of Simple and Quadratic Equations (1). Homogeneous Equations (3). Equations solved by changing the Unknowns (5). Reciprocal Equations (6).

CHAPTER II

Arithmetic Progression 12

General Form of an A.P. (13). Sum of n terms (17). Arithmetic Mean (21). Miscellaneous Theorems (23).

CHAPTER III

Geometric Progression 29

General Form of a G.P. (29). Geometric Mean (31). Sum of n terms (34). Infinite Series (36).

CHAPTER IV

Harmonic Progression 41

General Form of an H.P. (41). Harmonic Mean (42). Combination Series (47).

CHAPTER V

Ratio and Proportion 51

Definitions (51). Continued Proportion (52). Ratios of Greater or Less Inequality (55). Cross Multiplication (58). Elimination (61). Theorems in Ratio (63).

CHAPTER VI

PAGE
Variation 69
Direct and Inverse Variation (69). Variation with more than two Variables (74).

CHAPTER VII

Surds 79
Rationalizing Factor (79). Square Root of a Quadratic Surd (81). Roots of Unity (84).

CHAPTER VIII

Theory of Equations 89
Review of Primary Work (89). Variations in Sign of $ax^2 + bx + c$ (95). Condition for Rational Factors (98). The Remainder Theorem (99). Zero and Infinite Roots (101).

CHAPTER IX

Inequalities, Maxima and Minima 105
Quantities restricted in Value (105). Maximum and Minimum Values of Algebraic Functions (106).

CHAPTER X

Miscellaneous Series 113
Arithmetico-geometric Series (113). Squares and Cubes of the Natural Numbers (116). Mathematical Induction (119).

CHAPTER XI

PAGE
Permutations and Combinations 122
Fundamental Theorem (123). Value of ${}_nP_r$ (125). Permutations with Repetitions (127). Value of ${}_nC_r$ (133). Greatest value of ${}_nC_r$ (141). Miscellaneous Examples (144).

CHAPTER XII

Binomial Theorem. Positive Integral Indices . . 147
Law of Multiplication (147). Binomial Expansion (150). General Term (154). Properties of the Binomial Coefficients (157). Greatest Term (159). Miscellaneous Examples (162).

CHAPTER XIII

Binomial Theorem—Any Index 165
General Term (168). Greatest Term (172). Summation of Series (173). Approximations (177). Miscellaneous Examples (181).

CHAPTER XIV

Annuities 184
Review of Interest and Discount (184). Amount and Present Value of Annuity (187). Perpetual Annuity (190). Present Value of Mortgage, Bond or Debenture (192). Sinking Funds (193). Depreciation Funds (194). General Formulæ (197).

CHAPTER XV

Probability and Insurance 201
Simple Probability (201). Compound Probability (204). Mathematical Expectation (206). Mortality Table (206). Life Insurance Premiums (207).

CHAPTER XVI

PAGE

Scales of Notation 210

Simple Operations (210). Transferring from one Scale to another (213). Radix Fractions (215). Theorems on Divisibility (216).

Miscellaneous Examples 222

Tables of Interest, Discount, Logarithms and Mortality 236

Answers 241

SENIOR
HIGH SCHOOL ALGEBRA

CHAPTER I

EQUATIONS

In the Junior Algebra many equations have been solved, chiefly of the first or second degree. This chapter will furnish a review of the previous work, and will introduce some types not then considered.

1. Linear Equation. Every linear equation or equation of the first degree, in one unknown, may be reduced to the form :

$$ax = b.$$

The root of this equation is $x = \frac{b}{a}$.

2. Quadratic Equation. Every equation of the second degree, in one unknown, may be reduced to the form :

$$ax^2 + bx + c = 0.$$

The two roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3. Equations of a Higher Degree than the Second. When an equation of the third degree is reduced to the form :

$$ax^3 + bx^2 + cx + d = 0,$$

if the left-hand member can be factored, the equation can be completely solved.

Ex. 1.—Solve $x^3 - 4x^2 - 4x + 16 = 0$.

Since $x^3 - 4x^2 - 4x + 16 = x^2(x - 4) - 4(x - 4)$,

$$\therefore \quad (x - 4)(x^2 - 4) = 0,$$

$$\therefore \quad x - 4 = 0 \text{ or } x^2 - 4 = 0,$$

$$\therefore \quad x = 4 \text{ or } \pm 2.$$

Ex. 2.—Solve $3x^3 - 8x^2 + 6x - 1 = 0$.

Since the sum of the coefficients is zero, $x - 1$ is a factor,

$$\therefore \quad (x - 1)(3x^2 - 5x + 1) = 0,$$

$$\therefore \quad x = 1 \text{ or } \frac{5 \pm \sqrt{13}}{6}.$$

Similarly, when an equation of any degree is reduced to the form in which the right-hand member is zero, if the expression on the left can be expressed as the product of linear or quadratic factors, the equation can be completely solved.

4. Simultaneous Equations. Every pair of equations of the first degree, in two unknowns, may be reduced to the form :

$$ax + by = c,$$

$$a_1x + b_1y = c_1.$$

The roots are $x = \dfrac{b_1c - bc_1}{ab_1 - a_1b}$, $y = \dfrac{a_1c - ac_1}{a_1b - ab_1}$.

Every pair of equations in two unknowns, in which one is of the first degree and the other is of the second degree, may be reduced to the form :

$$ax + by = c,$$

$$mx^2 + nxy + py^2 + qx + ry = d.$$

Equations of this type may always be solved by finding the value of one of the unknowns in terms of the other from the linear equation and substituting this value in the other equation.

Ex.—Solve $2x - 3y = 3, \quad . \quad . \quad . \quad (1)$

$$x^2 + xy + y^2 + 2x - y = 18. \quad . \quad . \quad . \quad (2)$$

From (1) $$x = \frac{3 + 3y}{2},$$

$$\therefore \left(\frac{3 + 3y}{2}\right)^2 + y\left(\frac{3 + 3y}{2}\right) + y^2 + 3 + 3y - y = 18.$$

Simplifying, $$19y^2 + 32y - 51 = 0,$$

$$\therefore \quad y = 1 \text{ or } -\tfrac{51}{19},$$

$$x = 3 \text{ or } -\tfrac{48}{19}.$$

Two equations, each of which is of the second degree in two unknowns, cannot always be solved by elementary methods.

5. Homogeneous Equations. Equations of the following type in which the part involving the unknowns in each is homogeneous and of the second degree may always be solved:

$$ax^2 + bxy + cy^2 = d,$$

$$a_1x^2 + b_1xy + c_1y^2 = d_1.$$

Ex.—Solve $x^2 + xy - 2y^2 = 4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

$$2x^2 - xy - 3y^2 = 3. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If we multiply (1) by 3 and (2) by 4 and subtract the results, the absolute terms disappear and we get:

$$5x^2 - 7xy - 6y^2 = 0.$$

$$\therefore \quad x - 2y = 0 \text{ or } 5x + 3y = 0.$$

$$\therefore \quad x = 2y \text{ or } -\tfrac{3}{5}y.$$

Substitute $x = 2y$ in (1)

$\therefore \quad 4y^2 + 2y^2 - 2y^2 = 4,$

$\therefore \quad y = \pm 1.$

But $x = 2y,$

$\therefore \quad x = \pm 2.$

Substitute $x = -\tfrac{3}{5}y$ in (1)

$\therefore \quad \tfrac{9}{25}y^2 - \tfrac{3}{5}y^2 - 2y^2 = 4,$

$\therefore \quad y = \pm \tfrac{5}{14}\sqrt{-14}.$

But $x = -\tfrac{3}{5}y,$

$\therefore \quad x = \mp \tfrac{3}{14}\sqrt{-14}.$

6. Pairs of equations, in which one is of a higher degree than the second, may sometimes be solved by reducing them to simpler forms.

Ex. 1.—Solve $x^3 - y^3 = 63, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$

$x - y = 3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$

Divide the terms of (2) into the corresponding terms of (1) and we get :

$x^2 + xy + y^2 = 21. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

Now (2) and (3) may be solved as in Art. 4. Complete the solution. Solve also by substituting $x = y + 3$ in (1).

Ex. 2.—Solve $x^4 + x^2y^2 + y^4 = 91,$

$x^2 - xy + y^2 = 7.$

Divide as in the preceding example and then solve $x^2 + xy + y^2 = 13$, $x^2 - xy + y^2 = 7$, either by the method of Art. 5, or by finding the values of $x + y$ and $x - y$.

EXERCISE 1

Solve :

1. $\frac{2x-1}{3} - \frac{3x-2}{4} = \frac{5x-4}{6} - \frac{7x+6}{12}$.

2. $\frac{2}{2x-5} + \frac{1}{x-3} = \frac{6}{3x-1}$.

3. $\frac{x}{bc} + \frac{x}{ca} + \frac{x}{ab} = a + b + c.$

4. $ax + b = \frac{x}{a} + \frac{1}{b}$.

5. $780x^2 + 1 = 73x.$

6. $14x - x^2 = 33.$

7. $x^2 - 2ax + a^2 - b^2 = 0.$

8. $6x^3 - 5x^2 + x = 0.$

9. $x^3 - 1 = 0.$

10. $x^3 + 27 = 0.$

11. $x^4 - 1 = 0.$

12. $x^6 - 1 = 0.$

13. $\frac{1}{a+b+x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{x}$.

14. $\frac{a}{x-a} + \frac{b}{x-b} = \frac{2c}{x-c}$.

15. $2x^3 - 6x^2 - 3x + 7 = 0.$

16. $x^3 - nx - n + 1 = 0.$

17. $x^6 - 64x = 8x^4 - 8x^3.$

18. $x^6 - 243x = 9x^3(x - 3).$

19. $ax + by = c,$
$mx - ny = d.$

20. $\frac{x}{b+c} + \frac{y}{c+a} = \frac{2ax - 2by}{ac - bc} = 2.$

21. $x + y = 5,$
$3x^2 - y^2 = 23.$

22. $x - 2y + 1 = 0.$
$2x^2 + 3xy - 5y^2 - 7x - 2y + 9 = 0.$

23. $x^3 + y^3 = 351,$
$x^2 - xy + y^2 = 39.$

24. $8x^3 + y^3 = 91,$
$2x + y = 7.$

25. $5x^2 - 4xy - y^2 = 0$,
$x^2 + x + y = 5$.

26. $2y^2 - x^2 = 1$,
$3x^2 - 4xy = 7$.

27. $2x^2 + 3xy = 8$,
$y^2 - 2xy = 20$.

28. $x^2 + 2xy + 3y^2 = 17$,
$2x^2 + xy + y^2 = 8$.

29. $3x^2 + xy - 2y^2 = 16y$,
$x^2 - 2y^2 = 4y$.

30. $2x^3 - 2x^2y + xy^2 = 5$,
$y^3 + 5x^2y - 3xy^2 = 15$.

31. $x^4 + x^2y^2 + y^4 = 4251$,
$x^2 - xy + y^2 = 109$.

32. $x^2 + xy + y^2 = 21$,
$x + \sqrt{xy} + y = 7$.

7. Equations which may be solved by changing the Unknowns. There are certain equations whose solution may be more easily effected by making a change in the unknown or unknowns. Some types of these are illustrated in the following examples:

Ex. **1.**—Solve $x^2 + 6x + 2\sqrt{x^2 + 6x} = 24$.

If we attempt to remove the surd in the usual way by squaring, an equation of the fourth degree will result.

The solution is readily obtained if we first consider $\sqrt{x^2 + 6x}$ to be the unknown.

Let $\sqrt{x^2 + 6x} = y$, then $x^2 + 6x = y^2$,
$\therefore \quad y^2 + 2y = 24$ or $y^2 + 2y - 24 = 0$,
$\therefore \quad y = 4$ or -6.

We will assume that the radical sign indicates the positive square root only, so that $y = -6$ is discarded.

$\therefore \quad \sqrt{x^2 + 6x} = 4$, or $x^2 + 6x - 16 = 0$,
$\therefore \quad x = 2$ or -8.

Ex. **2.**—Solve $(x + 3)(x + 4)(x - 5)(x - 6) = 360$.

Multiply the first and third factors, and the second and fourth and we get

$$(x^2 - 2x - 15)(x^2 - 2x - 24) = 360.$$

Let $x^2 - 2x - 15 = y$, then $x^2 - 2x - 24 = y - 9$.

$\therefore \quad y(y - 9) = 360$,
$\therefore \quad y^2 - 9y - 360 = 0$,
$\therefore \quad y = 24$ or -15.

If $x^2 - 2x - 15 = 24$,
$x^2 - 2x - 39 = 0$,
$\therefore\ x = 1 \pm 2\sqrt{10}$.

If $x^2 - 2x - 15 = -15$,
$x^2 - 2x = 0$,
$\therefore\ x = 2$ or 0.

Ex. 3.—Solve $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$.

Divide by x^2 and $6x^2 + 5x - 38 + \frac{5}{x} + \frac{6}{x^2} = 0$,

$$\therefore\ 6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) = 38.$$

Let $x + \frac{1}{x} = y$, then $x^2 + \frac{1}{x^2} = y^2 - 2$,

$$\therefore\ 6(y^2 - 2) + 5y - 38 = 0,$$
$$\therefore\ 6y^2 + 5y - 50 = 0,$$
$$\therefore\ y = \tfrac{5}{2} \text{ or } -\tfrac{10}{3}.$$

If $y = \frac{5}{2}$, $x + \frac{1}{x} = \frac{5}{2}$,
$\therefore\ 2x^2 - 5x + 2 = 0$,
$\therefore\ x = 2$ or $\frac{1}{2}$.

If $y = -\frac{10}{3}$, $x + \frac{1}{x} = -\frac{10}{3}$,
$\therefore\ 3x^2 + 10x + 3 = 0$,
$\therefore\ x = -3$ or $-\frac{1}{3}$.

Here the four values of x are reciprocals in pairs. The reason for this is that the original equation is unchanged if we substitute $\frac{1}{x}$ for x.

Such an equation is called a **reciprocal equation.** This type may always be recognized by having the coefficients of the terms equidistant from the beginning and end the same.

Ex. 4.—Solve $x^2 + y^2 + 2x + 2y = 23$,
$x + y = \frac{5}{6}xy$.

Since $x^2 + y^2 = (x + y)^2 - 2xy$, we may consider $x + y$ and xy as the unknowns which should first be found.

Let $x + y = a$ and $xy = b$,

$\therefore\ a^2 - 2b + 2a = 23$ and $a = \frac{5}{6}b$.

Solving for a and b we get $a = 5$ or $-\frac{23}{6}$,
$b = 6$ or $-\frac{138}{25}$.

If $x + y = 5$, $xy = 6$,
$x = 3$ or 2,
$y = 2$ or 3.

If $x + y = -\frac{23}{5}$, $xy = -\frac{138}{25}$.
$x = \frac{1}{10}(-23 \pm \sqrt{1081})$,
$y = \frac{1}{10}(-23 \mp \sqrt{1081})$.

Ex. 5.—Solve $x + y = 4$, ... (1)
$x^4 + y^4 = 82$. . . . . (2)

Let $x = m + n$, $y = m - n$.
Substituting in (1), $2m = 4$ or $m = 2$.
Substituting in (2), $(m + n)^4 + (m - n)^4 = 82$.
Simplifying, $2m^4 + 12m^2n^2 + 2n^4 = 82$.
But $m = 2$ $\therefore\ 32 + 48n^2 + 2n^4 = 82$,
$\therefore\ n^4 + 24n^2 - 25 = 0$,
$\therefore\ n^2 = 1$ or -25,
$\therefore\ n = \pm 1$ or $\pm 5\sqrt{-1}$.
$\therefore\ x = m + n = 3$ or 1 or $2 \pm 5\sqrt{-1}$,
and $y = m - n = 1$ or 3 or $2 \mp 5\sqrt{-1}$.

EXERCISE 2

Solve and verify; reject extraneous roots:

1. $\sqrt{x+1} + \sqrt{2x+3} = 5$. **2.** $\sqrt{4x-6} = \sqrt{x+1} - \sqrt{3x-2}$.

3. $\sqrt{3x^2 - 2x + 9} - \sqrt{3x^2 - 2x - 15} = 2$.

4. $\sqrt{2x^2 - 7x + 1} - \sqrt{2x^2 - 9x + 4} = 1$.

5. $x^2 + 3 = 2\sqrt{x^2 - 2x + 2} + 2x$. **6.** $x^2 - 3x - 6\sqrt{x^2 - 3x + 2} = -2$.

7. $2x^2 + 6x - 17 = \sqrt{x(x+3) - 1}$.

8. $10x + 5\sqrt{3x^2 - 5x + 2} = 6x^2 + 6.$

9. $\sqrt{19 + x} - \sqrt{15 - x} = \dfrac{24}{\sqrt{19 + x}}.$

10. $\dfrac{3x - 2}{2} + \sqrt{2x^2 - 5x + 3} = \dfrac{(x + 1)^2}{3}.$

11. $x^4 + 100 = 29x^2.$

12. $(x^2+x+3)(x^2+x+5)=35.$

13. $\dfrac{x}{x^2 + 1} + \dfrac{x^2 + 1}{x} = 2\frac{9}{10}.$

14. $x + \dfrac{1}{x} + \left(x + \dfrac{1}{x}\right)^{-1} = 2\frac{9}{10}.$

15. $(x - 2)(x - 3)(2x - 3)(2x - 5) = 210.$

16. $(4x^2 + 16x + 15)(x^2 + 7x + 12) = 700.$

17. $6x^4 - 7x^3 - x^2 - 7x + 6 = 0.$

18. $12(x^4+1)-13x(x^2+1)=x^2.$

19. $x^4-10x^3+26x^2-10x+1=0.$

20. $6x^4-25x^3+12x^2+25x+6=0.$

21. $xy + x + y = 11,$
$x^2y + xy^2 = 30.$

22. $(x + y)^2 - 189 = xy,$
$x + y - 21 = -\sqrt{xy}.$

23. $x^2 + y^2 + x - y = 6,$
$xy + x - y = 3.$

24. $x^2y + xy^2 = 290,$
$x^2 + y^2 = 29.$

25. $x - y - \sqrt{x - y} = 2,$
$x^3 - y^3 = 2044.$

26. $x^2 + y^2 = 41,$
$x + y + xy = 29.$

27. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{3},$
$x^2 + y^2 = 160.$

28. $\dfrac{1}{x} + \dfrac{1}{y} = 15,$
$\dfrac{1}{x^3} + \dfrac{1}{y^3} = 1125.$

29. $x - y = 19,$
$x^{\frac{1}{3}} - y^{\frac{1}{3}} = 1.$

30. $x^2 + 9y^2 + 3x - 9y = 28,$
$xy = 4.$

31. $x^2 + xy + y^2 - 3x - 3y = 16,$
$(x - 2)(y - 2) = 2.$

32. $x^3 + y^3 = 28,$
$x^2y + xy^2 = 12.$

33. $x^4 + y^4 = 1312,\ x + y = 8.$

34. $x^4 + y^4 = 626,\ x - y = 4.$

35. Find four consecutive integers whose product is 17160.

36. Find four consecutive odd integers whose product is 945.

37. Find the five fifth roots of unity.

8. Equations with three Unknowns. Any three equations of the first degree with three unknowns may be reduced to the form

$$\begin{aligned} ax + by + cz &= d, \\ a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2. \end{aligned}$$

If we eliminate z from two pairs of these equations and then eliminate y from the two resulting equations we will find that

$$x = \frac{d(b_1c_2 - b_2c_1) + d_1(b_2c - bc_2) + d_2(bc_1 - b_1c)}{a(b_1c_2 - b_2c_1) + a_1(b_2c - bc_2) + a_2(bc_1 - b_1c)}.$$

The values of y and z may then be written down by symmetry from the value of x.

If two of the equations are of the first degree and the other of the second degree, the equations can always be solved.

Ex. 1.—Solve

$$\begin{aligned} x + y - 2z &= 3, \quad \ldots\ldots (1) \\ 2x + y - 3z &= 5, \quad \ldots\ldots (2) \\ x^2 + y^2 - z^2 &= 12. \quad \ldots\ldots (3) \end{aligned}$$

Eliminate y from (1) and (2) and $x - z = 2$ or $x = z + 2$.

Eliminate x from (1) and (2) and $y - z = 1$ or $y = z + 1$.

Substitute $x = z + 2$ and $y = z + 1$ in (3) and

$$(z + 2)^2 + (z + 1)^2 - z^2 = 12.$$

$$\therefore \quad z = 1 \text{ or } -7,$$

$$x = 3 \text{ or } -5,$$

$$y = 2 \text{ or } -6.$$

$\therefore$ the values of x, y, z are $(3, 2, 1)$ or $(-5, -6, -7)$.

When two of the three equations are of a higher degree than the first, they can be solved by elementary methods in some special cases.

Ex. 2.—Solve
$$x + 2y + z = 12,$$
$$x^2 + 4y^2 + z^2 = 56,$$
$$xy = 6.$$

Since $x^2 + 4y^2 = (x + 2y)^2 - 4xy$, we might consider $x + 2y$, xy and z as the unknowns, and if we let $x + 2y = a$ and $xy = b$, then the equations are
$$a + z = 12,$$
$$a^2 - 4b + z^2 = 56,$$
$$b = 6.$$

From these we easily obtain $a = 8$ or 4, $z = 4$ or 8, $b = 6$.

If $x + 2y = 8$, $xy = 6$, $z = 4$,
then $x = 2$ or 6,
$y = 3$ or 1,
$z = 4$.

If $x + 2y = 4$, $xy = 6$, $z = 8$,
then $x = 2 \pm 2\sqrt{-2}$,
$y = 1 \mp \sqrt{-2}$,
$z = 8$.

The four sets of values of x, y, z are
$$(2, 3, 4), (6, 1, 4), (2 \pm 2\sqrt{-2}, 1 \mp \sqrt{-2}, 8).$$

EXERCISE 3

Solve :

1. $2x + 3y + 6z = 8$, $4x - 5y - z = 14$, $5x - 8y - 2z = 4$.

2. $\frac{x}{2} + \frac{y}{3} = \frac{y}{3} + \frac{z}{4} = \frac{z}{4} + \frac{x}{2} = 1$. **3.** $\frac{1}{x} + \frac{1}{y} = \frac{3}{y} + \frac{3}{z} = \frac{2}{z} + \frac{2}{x} = 1$.

4. $x + y + z = a + b + c$, $bx + cy + az = cx + ay + bz = a^2 + b^2 + c^2$.

5. $x + y - z = 0$, $3x - 4y + z = 0$, $x^2 + y^2 + z^2 = 6x + 3y + z$.

6. $3x - 2y - z = x - y + 2z = 0$, $x^2 + y^2 + z^2 = 75$.

7. $3x - 2y - 3z = 0$, $x - 10y + 6z = 0$, $x^2 + y^2 - z^2 = 116$.

8. $x - y + 2z = 4$, $3x + y - z = 13$, $xy + yz + zx = 14$.

9. $x + y + z = 7$, $x^2 + y^2 + z^2 = 21$, $yz = 8$.

10. $x + y + z = 14$, $x^2 + y^2 + z^2 = 84$, $xy = z^2$.

11. $x(y + z) = 17$, $y(z + x) = 20$, $z(x + y) = 5$.

12. $x^2 + xy + xz = 30$, $y^2 + yz + xy = 20$, $z^2 + xz + yz = 50$.

13. $3(x + y)(y + z) = 5(y + z)(z + x) = 4(z + x)(x + y) = 60$.

14. $(x + y)(x + z) = 4bc$, $(y + x)(y + z) = 4ac$, $(z + x)(z + y) = 4ab$.

15. $(x + y + z)(y + z) = 70$, $(x + y + z)(z + x) = 40$,
$(x + y + z)(x + y) = 90$.

16. $x + y + z = a + b + c$, $ax + by + cz = bc + ca + ab$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

17. $x + y - z = 4$, $x^2 + y^2 - z^2 = 12$, $x^3 + y^3 - z^3 = 34$.

18. $xy = 2(x + y) + 8$, $yz = 2(y + z) + 16$, $zx = 2(z + x) + 11$.

19. $x(y + z - x) = -7$, $y(z + x - y) = 2$, $z(x + y - z) = -12$.

20. $x + yz = y + zx = z + xy = 6$.

21. $x^3 + y^3 + z^3 = 3xyz$, $x - a = y - b = z - c$.

CHAPTER II

ARITHMETIC PROGRESSION

9. Series. A sequence of quantities in which each is formed from one or more of the preceding by a definite law is called a **series.** The successive quantities are called the **terms** of the series.

Consider the following series :

(1) 2, 5, 8, **11** . . .
(2) 13, 11, 9, 7 . . .
(3) 3, 6, 12, 24 . .
(4) 81, 27, 9, 3 . . .

What is the law of formation in each of these series ? What would be the next four terms in each ?

10. Arithmetic Progression. A series in which each term, after the first, is formed by adding a constant quantity to the preceding term is called an **arithmetic series or progression (A.P.).**

The constant quantity is called the **common difference** of the series and is found by subtracting any term from the term which *follows* it.

The following are examples of arithmetic series :

(1) 1, 3, 5, 7, . . .
(2) 15, 5, − 5, − 15, . . .
(3) $7\frac{1}{2}$, 0, $-7\frac{1}{2}$, . . .
(4) a, $(a + d)$, $(a + 2d)$, . . .

The common difference in (1) is 2, in (2) is − 10, in (3) is $-7\frac{1}{2}$, and in (4) is d.

11. Any Term of an A.P. When the first two terms, or the first term and the common difference, of an A.P. are given any term may be written down.

Thus, in the series 4, 7, 10, 13, . . . , the 3rd term is found by adding the common difference (3) twice to the 1st term, the 4th by adding the common difference three times, and so on.

That is, the 3rd term $= 4 + 2 \times 3 = 10$,
the 4th term $= 4 + 3 \times 3 = 13$.
Similarly, the 20th term $= 4 + 19 \times 3 = 61$,
and the 47th term $= 4 + 46 \times 3 = 142$.

12. General Form of an A.P. If we let a be the 1st term and d the common difference, the general form of an A.P. is

$$a, (a + d), (a + 2d), (a + 3d), \ldots$$

Since a and d may have any values, this series represents every arithmetic series.

The 3rd term $= a + 2d$,
The 4th term $= a + 3d$,
The 10th term $= a + 9d$.

That is, the coefficient of d in each term is one less than the number of the term,

$$\therefore \text{ the } n\text{th term} = a + (n - 1)d.$$

If the series contains n terms, and we represent the last term by l, we have the formula.

$$\boldsymbol{l = a + (n - 1)d}.$$

Any term of an A.P. may be found by substituting in this formula the particular values of a, d and n.

Ex. 1.—Find the 7th, 20th, and nth terms of

(1) 2, 7, 12, 17, . . . (2) 15, $12\frac{1}{2}$, 10, . . .

In (1) $a = 2$ and $d = 5$,

$\therefore$ the 7th term $= 2 + 6 \times 5 = 32$,
the 20th term $= 2 + 19 \times 5 = 97$,
the nth term $= 2 + (n - 1)5 = 5n - 3$.

In (2) $a = 15$ and $d = -2\frac{1}{2}$,

$\therefore$ the 7th term $= 15 + 6 \times (-2\frac{1}{2}) = 0$,
the 20th term $= 15 - 19 \times 2\frac{1}{2} = -32\frac{1}{2}$,
the nth term $= 15 - (n - 1)2\frac{1}{2} = 17\frac{1}{2} - 2\frac{1}{2}n$.

Ex. 2.—Which term of 8, 11, 14 . . . is 68?

Suppose 68 is the nth term, then since $a = 8$, $d = 3$,

$$68 = 8 + (n - 1)3,$$
$$\therefore \quad n = 21.$$

$\therefore$ the 21st term is 68.

Verify this by finding the 21st term.

13. If any two Terms of an A.P. are given, the Series may be completely determined. For, from the data two equations in a and d may be formed, from which the values of a and d may be found.

Ex.—Determine the A.P. in which the 3rd term is 10 and the 14th term is 54, and find the nth term.

Let the series be a, $a + d$, $a + 2d$, . . .

Then $$a + 2d = 10,$$

and $$a + 13d = 54.$$

Solving we find $d = 4$ and $a = 2$,

$\therefore$ the series is 2, 6, 10, 14, . . .

The nth term is $2 + (n - 1)4 = 4n - 2$.

Verify by substituting $n = 3$ and $n = 14$ in this nth term and see if the results agree with the conditions of the problem.

Generally, if the pth term is m, and the qth term is n the values of a and d may be found by solving

$$a + (p - 1)d = m,$$
$$a + (q - 1)d = n.$$

EXERCISE 4

Find the 5th and 12th terms of the following series :

1. 3, 8, 13, . . . **2.** 11, 6, 1, . . . **3.** $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, . . .

4. $3\frac{1}{2}$, 2, $\frac{1}{2}$, . . . **5.** a, $3a$, $5a$, . . . **6.** 1·2, ·8, ·4, . . .

7. $a + b$, a, $a - b$, . . . **8.** $2a - b$, $3a - 2b$, $4a - 3b$, . . .

9. $x + 1$, $2x + 3$, $3x + 5$, . . . **10.** $m + n$, 0, $- m - n$, . . .

11. Find the 100th term of 1, 3, 5, . . . and the 50th term of $7\frac{3}{4}$, $9\frac{1}{2}$, $11\frac{1}{4}$, . . .

12. Find the 20th term of $\frac{1}{\sqrt{2}}$, $\sqrt{2}$, $\frac{3}{\sqrt{2}}$, . . . and the 23rd term of $\frac{3}{\sqrt{5}}$, $\frac{4}{\sqrt{5}}$, $\sqrt{5}$, . . .

Find the nth term of the following series :

13. 1, 2, 3, . . . **14.** 1, 3, 5, . . . **15.** 7, 10, 13, . . .

16. 2, − 1, − 4, . . . **17.** $3x$, x, $- x$, . . . **18.** $2n - 1$, $2n - 3$, $2n - 5$, . . .

19. $(a + b)^2$, $a^2 + b^2$, $(a - b)^2$, . . . **20.** $\frac{n - 1}{n}$, $\frac{n - 2}{n}$, $\frac{n - 3}{n}$, . . .

Which term of the series

21. 1, 3, 5, . . . is 39 ? **22.** 9, 6, 3, . . . is − 288 ?

23. 21, $19\frac{1}{2}$, 18, . . . is 0 ? **24.** $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, . . . is − 3 ?

25. If the first term of an A.P. is 3 and the 20th term is 79, what is the common difference ?

26. If the 8th term of an A.P. is 50 and the 21st term is 115, find the 50th term.

27. If the first term of an A.P. is 2 and the last term is 119, find the number of terms, the common difference being 3.

28. Find the first term of an A.P. in which the 15th term is − 40 and the 31st term is 40.

29. If the 3rd term of an A.P. is 9 and the 10th term is − 12, which term is − 60 ?

30. How many odd numbers are there from 13 to 193, both inclusive?

31. How many multiples of 7 are there between 100 and 500?

32. The sum of the 2nd and 5th terms of an A.P. is 26 and of the 4th and 20th is 94. Find a and d.

33. The sum of the first three terms of an A.P. is less than the sum of the next three by 54. Find d.

34. Find the 20th term from the end of the series 3, 7, 11, . . . , 199. (Reading from the end $a = 199$, $d = -4$.)

35. Find the sum of the 8th term from the beginning and the 8th term from the end of 62, 59, 56, . . . , 2.

36. In the series a, $a + d$, . . . , $l - d$, l, show that the sum of the rth term from the beginning and the rth term from the end is constant for all values of r.

37. If an A.P. has an odd number of terms, show that the sum of the first and last terms is twice the middle term.

38. The sum of three numbers in A.P. is 30. Find the middle number.

39. If the sum of three numbers in A.P. is 15 and the sum of their squares is 83, find the numbers. (Let the numbers be $a - d$, a, $a + d$.)

40. If the 10th, 20th, and nth terms of an A.P. are 29, 59 and 137 respectively, find n.

41. If $x + y$, $2x - 2y$, $5x - 6y$ are in A.P., show that $y = 2x$.

42. If a, b, c, d are four consecutive terms of an A.P., show that $a^2 - 3b^2 + 3c^2 - d^2 = 0$.

43. The sides of a right-angled triangle are in A.P. If the hypotenuse is 25, find the other sides.

44. Show that the $(n - r + 1)$th, $(n + 1)$th and $(n + r + 1)$th terms of any A.P. are in A.P.

45. Find the sum of the first 50 odd numbers by adding the first and last, the second and next to last, and so on.

14. Sum of an A.P. Suppose it is required to find the sum (S) of 8 terms of the series 3, 8, 13, . . .

$$S = 3 + 8 + 13 + 18 + 23 + 28 + 33 + 38,$$

or $$S = 38 + 33 + 28 + 23 + 18 + 13 + 8 + 3.$$

Adding on each side we get

$$2S = 41 + 41 + 41 + 41 + 41 + 41 + 41 + 41,$$
$$= 41 \times 8,$$
$$\therefore \quad S = (41 \times 8) \div 2 = 164.$$

Similarly, if S is the sum of n terms of

$a, a + d, a + 2d, \ldots$ and if l denotes the last term,

then $S = a + (a+d) + (a+2d) + \ldots + (l-2d) + (l-d) + l,$

or $S = l + (l-d) + (l-2d) + \ldots + (a+2d) + (a+d) + a.$

$\therefore \quad 2S = (a + l) + (a + l) + (a + l) + \ldots$ to n terms,

$$= n(a + l).$$

$$\therefore \quad \boldsymbol{S = \frac{n}{2}(a + l).} \quad \ldots\ldots \quad (1)$$

But $$\boldsymbol{l = a + (n - 1)d,} \quad \ldots\ldots \quad (2)$$

$$\therefore \quad \boldsymbol{S = \frac{n}{2}\{2a + (n - 1)d\}.} \quad \ldots \quad (3)$$

These three formulæ are important and they should be committed to memory.

Ex. 1.—Find the sum of 50 terms of $3 + 5 + 7 + \ldots$

Here $$a = 3, d = 2, n = 50,$$

$$\therefore \quad S = \frac{n}{2}\{2a + (n - 1)d\} = \frac{50}{2}(6 + 49 \times 2) = 2600.$$

Ex. 2.—Find the sum of all multiples of 3 between 100 and 500.

Here the first term is 102 and the last is 498.
If we suppose n to be the number of terms,

$$\therefore \quad \text{the } n\text{th term} = 102 + (n - 1)3 = 498,$$
$$\therefore \quad n = 133.$$

$$\therefore \quad S = \frac{n}{2}(a + l) = \frac{133}{2}(102 + 498) = 39900.$$

15. Each of the three formulæ in Art. 14 contains four letters, and consequently if the values of any three be given the other can be found.

Thus, in (1) if a, n, l are given, $S = \frac{n}{2}(a + l)$,

if a, l, S are given, $n = \frac{2S}{a + l}$,

if a, n, S are given, $l = \frac{2S - an}{n}$.

In (2) find the value of each letter in terms of the other three letters.

In (3) when the brackets are removed we get

$$2S = 2an + n^2d - nd.$$

Here if a, d and S are given, and we wish to find n, a quadratic equation will result and therefore there will be two values of n.

Ex. **1.**—How many terms of the series

$$21 + 18 + 15 + \ldots \text{ will give the sum } 81?$$

Here $a = 21, d = -3, S = 81,$

$$\therefore \quad 81 = \frac{n}{2}\{42 - (n - 1)3\},$$

$$n^2 - 15n + 54 = 0,$$

$$\therefore \quad n = 6 \text{ or } 9.$$

Therefore the number of terms is either 6 or 9.

Write down 9 terms of the series and discover why the sum of 6 terms is the same as the sum of 9 terms.

Here each root of the equation gives an admissible answer to the given problem, but naturally both of the values of n would not always be integral and positive.

Ex. **2.**—How many terms of $40 + 33 + 26 + \ldots$ must be taken so that the sum may be 99?

Here $99 = \frac{n}{2}\{80 - (n - 1)7\}$.

Solving we get $n = 3$ or $9\frac{3}{7}$.

Here the first answer 3 is admissible for

$$40 + 33 + 26 = 99.$$

The value $n = 9\frac{3}{7}$ does not furnish an intelligible solution. Write down 10 terms of the series and compare their sum with the sum of 9 terms. What conclusion do you draw?

Ex. 3.—How many terms of $12 + 16 + 20 \ldots$ will make 208?

Solving as before we find $n = 8$ or -13.

Write down 8 terms, the last of which is 40. If now we find the sum of 13 terms counting backwards from 40 the sum will also be 208. Verify this.

EXERCISE 5

Find the sum of the arithmetic series whose first and last terms and number of terms are:

1. 1, 79; 40. **2.** 5, 75; 10. **3.** 25, -50; 32.

4. a, $25a$; 13. **5.** $-3x$, $7x$; n. **6.** $(a + b)$, $(a - b)$; $2n$.

Find the sum of:

7. 1, 4, 7, . . . to 20 terms. **8.** 9, 13, 17, . . . to 12 terms.

9. 12, 6, 0, . . . to 32 terms. **10.** 2, $3\frac{1}{2}$, 5, . . . to 18 terms.

11. -11, $-9\frac{1}{2}$, -8, . . . to 15 terms.

12. 99, 97, 95, . . . to 99 terms.

13. $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$, . . . to 16 terms. **14.** 1, $\frac{1}{3}$, $-\frac{1}{3}$, . . . to 18 terms.

15. a, $3a$, $5a$, . . . to 12 terms. **16.** 1, 1·1, 1·2, . . . to 100 terms.

Find the sum of n terms of:

17. $2 + 5 + 8 + \ldots$ **18.** $1 + 2 + 3 + \ldots$

19. $2 + 6 + 10 + \ldots$ **20.** $5 + 11 + 17 + \ldots$

21. $3 + 1 - 1 - \ldots$ **22.** $4 + 1\frac{1}{2} - 1 - \ldots$

23. $a + 5a + 9a + \ldots$ **24.** $(a + 1) + (a + 3) + (a + 5) + \ldots$

25. $\left(1 - \frac{1}{n}\right) + \left(1 - \frac{2}{n}\right) + \left(1 - \frac{3}{n}\right) + \ldots$

26. $(a + b)^2 + (a^2 + b^2) + (a - b)^2 + \ldots$

27. Find the sum of the first n odd numbers.

28. Show that $3(1+3+5+ \ldots +99)=101+103+105 \ldots +199$.

29. Find the sum of 20 consecutive odd numbers of which the greatest is 125.

30. If the 5th term of an A.P. is 8 and the 10th term is -2, find the sum of 17 terms.

31. Show that $1 + \frac{n-3}{n-1} + \frac{n-5}{n-1} + \frac{n-7}{n-1} + \ldots$ to $(n-1)$ terms $= 1$.

32. If the sum of $16 + 14 + 12 + \ldots$ is 72, find the number of terms.

33. If $a = 4\frac{1}{2}$, $d = -\frac{1}{4}$ and $S = 41\frac{1}{4}$, find n.

34. If $3 + 3\frac{1}{8} + 3\frac{1}{4} + \ldots$ to n terms $= 68$, find n.

35. If the sum of the 4th and 8th terms of an A.P. is 24, and the sum of the 15th and 19th is 68, find the sum of n terms.

36. Find the A.P. in which the sum of the first 10 terms is 100 and the sum of the next 10 terms is 300.

37. A travels 2 miles the first hour, $2\frac{1}{4}$ the second, $2\frac{1}{2}$ the third, and so on. B travels 4 miles per hour. In how many hours will A overtake B, if they start together?

38. There are 40 stones in a row, 1 yard apart. How far does a boy travel in bringing them together, one by one, at the first stone?

39. Show that the sum of $2n + 1$ terms of an A.P. is $2n + 1$ times the middle term.

40. Find the sum of all multiples of 6 between 1000 and 2000.

41. Find the sum of all odd numbers with three digits.

42. The first term of an A.P. is 1 and the sum of n terms is n^3, find the common difference.

43. Find the sum of all the integers from 10 to 100, both inclusive, which are not divisible by 3.

44. A body falling freely falls 16·1 feet in the first second, and in each succeeding second 32·2 feet more than in the preceding second. If a stone dropped from a stationary balloon reaches the ground in 12 seconds, how high is the balloon?

45. A tower is 1030·4 feet high. In how many seconds would a stone dropped from the top of it reached the ground?

46. The sum of n terms of the series, 1, 4, 7, . . . is to the sum of $2n$ terms as 10 : 41; find n.

47. If the second term of an A.P. is 28, the sixth is 12, and the last is − 28, find the sum of all the terms.

48. With which term must I begin in the series 99, 97, 95, . . . so that the sum of 40 consecutive terms will be 960?

16. Arithmetic Mean. When three terms are in A.P. the middle one is called the **arithmetic mean** between the other two.

Thus, 7 is the arithmetic mean between 3 and 11 since 3, 7, 11 are in A.P.

If x is the arithmetic mean between a and b, then a, x, b are in A.P.,

$$\therefore \quad x - a = b - x,$$

$$\therefore \quad x = \frac{a + b}{2}.$$

Therefore *the arithmetic mean between any two quantities is one-half of their sum.*

17. When any number of terms form an A.P., the terms intermediate between any two are called the **arithmetic means** between those two terms.

Thus, 3, 5, 7 are three arithmetic means between 1 and 9 since 1, 3, 5, 7, 9 are in A.P.

Ex.—Insert 4 arithmetic means between 5 and 85.

The whole series consists of 6 terms of which the first is 5 and the 6th is 85.

$\therefore \quad 85 = 5 + 5d,$

$\therefore \quad d = 16,$

$\therefore$ the means are 21, 37, 53, 69.

To insert n arithmetic means between a and b. The whole series consists of $n + 2$ terms of which the first is a and the last or $(n + 2)$th is b.

Therefore if d is the common difference

$$a + (n + 1)d = b, \text{ or } d = \frac{b - a}{n + 1},$$

and the required means are

$$a + \frac{b - a}{n + 1},\ a + \frac{2(b - a)}{n + 1},\ a + \frac{3(b - a)}{n + 1}, \ldots,\ a + \frac{n(b - a)}{n + 1},$$

or $$\frac{na + b}{n + 1},\ \frac{(n - 1)a + 2b}{n + 1},\ \frac{(n - 2)a + 3b}{n + 1}, \ldots,\ \frac{a + nb}{n + 1}.$$

EXERCISE 6

1. What is the arithmetic mean between 16 and 20; -11 and -41; $a + b$ and $a - b$; $5x - 3y$ and $3x + 5y$?

2. Insert 6 arithmetic means between 11 and 53; 8 between 5 and 11; 10 between 13 and $29\frac{1}{2}$; 12 between $4a + 3b$ and $3b - 9a$; x between 1 and x^2.

3. Insert 20 arithmetic means between 2 and 65 and find their sum.

4. If 100 arithmetic means be inserted between -12 and 493, find the 65th mean.

5. When m arithmetic means are inserted between x and y, find the rth mean.

6. For what values of x is the arithmetic mean between $ax^2 + bx$ and $ax + bx^2$ equal to $a + b$?

7. If n arithmetic means be inserted between a and b, find their sum.

8. If an odd number of arithmetic means be inserted between a and b, show that the middle mean is the arithmetic mean between a and b.

9. If a, b, c are in A.P. and if x is the arithmetic mean between a and b and y between b and c, show that a, x, b, y, c are in A.P.

10. If a, x, y, b are in A.P., find x and y in terms of a and b.

11. If b is the arithmetic mean between a and c, show that $b^2(c + a)$ is the arithmetic mean between $a^2(b + c)$ and $c^2(a + b)$.

12. Between two numbers whose sum is $3\frac{1}{2}$, a number of arithmetic means is inserted. If the sum of the means exceeds their number by 9, find the number of means.

18. When the nth term of a series is given as a function of n, any term may be found by substituting a particular value for n.

Thus, if the nth term is $3n + 5$,
the 6th term is $3 \times 6 + 5 = 23$,
the 1st term is $3 \times 1 + 5 = 8$,
the 2nd term is $3 \times 2 + 5 = 11$.

If the series was known to be an A.P., then the series must be $8 + 11 + 14 + \ldots$

We might show that the series whose nth term is $3n + 5$ *must be an A.P.*

If the nth term is $3n + 5$,

$\therefore$ the $(n - 1)$th term is $3(n - 1) + 5 = 3n + 2$.

The difference between the nth and $(n - 1)$th terms is 3, which is a constant for all values of n. But the nth and $(n - 1)$th terms are *any* two consecutive terms, and therefore the difference between any two consecutive terms is 3. Then the series must be an A.P.

Or we might reason thus :

The nth term of the general A.P. is $a + (n - 1)d$ or $nd + (a - d)$. This is of the first degree in n and therefore

$3n + 5$, which is of the same form, must be the nth term of an A.P. in which

$$d = 3 \text{ and } a - d = 5,$$

or $$d = 3 \text{ and } a = 8,$$

$$\therefore \text{ the series is } 8 + 11 + 14 + \ldots$$

Similarly, the sum of n terms of the general A.P. is

$$\frac{n}{2}\left\{2a + (n-1)d\right\} \quad \text{or} \quad \frac{d}{2}n^2 + \left(a - \frac{d}{2}\right)n.$$

This is of the second degree in n, there being no absolute term. Any quantity of the same form as this would be the sum of n term of an A.P.

Thus, if the sum of n terms is $2n^2 + 7n$,

then $$\frac{d}{2} = 2 \text{ and } a - \frac{d}{2} = 7, \qquad \therefore \quad d = 4 \text{ and } a = 9,$$

$$\therefore \text{ the series is } 9 + 13 + 17 + \ldots$$

Verify this by finding the sum of n terms.

If the sum of n terms is given, the nth term may at once be found by subtracting the sum of $n - 1$ terms from the sum of n terms.

Thus, if the sum of n terms $= 5n^2 - 4n$,

then the sum of $n - 1$ terms $= 5(n-1)^2 - 4(n-1)$,

$$= 5n^2 - 14n + 9,$$

$$\therefore \text{ the } n\text{th term} = 10n - 9.$$

Ex. 1.—If S_n be the sum of n terms of an A.P. prove

$$S_{n+2} - 2S_{n+1} + S_n = d.$$

$S_{n+2} - S_{n+1}$ = the $(n + 2)$th term,

and $S_{n+1} - S_n$ = the $(n + 1)$th term,

$\therefore$ by subtraction, $S_{n+2} - 2S_{n+1} + S_n$

= the $(n + 2)$th term − the $(n + 1)$th term

$= d$.

We might also have proven it by substituting the values for S_{n+2}, S_{n+1}, S_n.

Ex. 2.—If the pth term of an A.P. is m and the qth term is n, find the $(p+q)$th term.

$$\text{The } p\text{th term} = a + (p-1)d = m.$$
$$\text{The } q\text{th term} = a + (q-1)d = n$$
$$\therefore \quad d = \frac{m-n}{p-q}.$$

$$\text{Then the } (p+q)\text{th term} = a + (p+q-1)d,$$
$$= a + (p-1)d + qd,$$
$$= m + qd = m + \frac{q(m-n)}{p-q},$$
$$= \frac{mp - nq}{p-q}.$$

It will be seen that it is not necessary to solve for a.

Ex. 3.—If a, b, c are in A.P. then the following are also in A.P.

(1) $a + x$, $b + x$, $c + x$.

(2) $a - x$, $b - x$, $c - x$.

(3) ax, bx, cx.

(4) $\frac{a}{x}$, $\frac{b}{x}$, $\frac{c}{x}$.

The proof of these is left as an exercise for the pupil.

Ex. 4.—If $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A.P., show that

$\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are also in A.P.

$\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A.P.

Multiply by $a+b+c$, $\therefore$ $\frac{a+b+c}{a}$, $\frac{a+b+c}{b}$, $\frac{a+b+c}{c}$ are in A.P.

Subtract 1 from each, $\therefore$ $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A.P.

Ex. 5.—Find the nth term and the sum of n terms of $1 + (2 + 3 + 4) + (5 + 6 + 7 + 8 + 9) + \ldots$

In n terms the number of natural numbers is

$$1 + 3 + 5 + \ldots \text{ to } n \text{ terms} = n^2.$$

$$\therefore \quad S = 1 + 2 + 3 + 4 + 5 + \ldots \text{ to } n^2 \text{ terms}$$

$$= \frac{n^2}{2}(2 + n^2 - 1) = \frac{n^2(n^2 + 1)}{2},$$

Since the sum of n terms $= \dfrac{n^2(n^2 + 1)}{2}$

$$\therefore \quad \text{the sum of } n - 1 \text{ terms} = \frac{(n - 1)^2\{(n - 1)^2 + 1\}}{2},$$

$$\therefore \quad \text{the } n\text{th term} = 2n^3 - 3n^2 + 3n - 1.$$

EXERCISE 7

1. If a, b, c are in A.P., then $a + b$, $a + c$, $b + c$ are in A.P.

2. Show that the series whose nth term is $2n - 3$ is an A.P. and find its sum to n terms.

3. The sum of n terms of a series is $2n^2 + 3n$. Find its nth term and show that the series is an A.P.

4. Find the series whose sum to x terms is $ax^2 - bx$.

5. Sum to n terms the series whose rth term is $3r + 2$.

6. If a, b, c, d are in A.P., prove that $bc > ad$.

7. Show that the series formed by taking every rth term of an A.P. is also an A.P.

8. Show that the sums of successive groups of n terms of an A.P. are also in A.P.

9. Find three numbers in A.P. whose sum is 27 and product 693.

10. If a, b, c are in A.P., show that $a^2 - bc$, $b^2 - ca$, $c^2 - ab$ are in A.P.

11. Show that the sum of the first n odd numbers is one-third the sum of the next n.

12. If a, b, c are in A.P., show that $(b + c)^2 - a^2$, $(c + a)^2 - b^2$, $(a + b)^2 - c^2$ are also in A.P.

13. If a, x, y, z, b are in A.P., find x, y and z in terms of a and b.

14. The 1st term of an A.P. is 27, the 4th term is 18 and the sum of all the terms is 117; find the last term.

15. Divide $\frac{n}{12}(n + 3)$ into n parts such that each shall exceed the preceding by a fixed quantity.

16. If a, b, c are in A.P., show that

$$\tfrac{2}{3}(a + b + c)^3 = a^2(b + c) + b^2(c + a) + c^2(a + b).$$

17. Find the least value of n for which the sum of the first n natural numbers exceeds 1000. Verify your result.

18. Find the sum of $2n$ terms and of $2n + 1$ terms of the series

$$1 - 3 + 5 - 7 + 9 - 11 + \ldots$$

19. Prove that the squares of $x^2 - 2x - 1$, $x^2 + 1$ and $x^2 + 2x - 1$ are in A.P.

20. The 1st, 2nd and last terms of an A.P. are a, b, c respectively, find its sum.

21. Divide 1 into four parts in A.P. so that the sum of their cubes may be $\frac{1}{10}$. (Let the parts be $a - 3d$, $a - d$, $a + d$, $a + 3d$.)

22. The sum of 5 numbers in A.P. is 10 and the sum of their squares is 60. Find the numbers.

23. If S_n denote the sum of n terms of an A.P., prove that

(1) $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n = 0.$

(2) $S_{3n} = 3(S_{2n} - S_n).$

(3) $S_{n+2m} - 2S_{n+m} + S_n = m^2d.$

24. Find four numbers in A.P. such that the sum of their squares shall be 120 and the product of the first and last shall be less than the product of the other two by 8.

25. If S_a represent the sum of n of the natural numbers beginning with a, prove that $S_{3a+n-1} = 3S_a$.

26. If the pth, qth, rth terms of an A.P. are x, y, z respectively, show that $x(q - r) + y(r - p) + z(p - q) = 0$.

27. Find the sum of n terms of

(1) $1 + (3 + 5) + (7 + 9 + 11) + \ldots$

(2) $1 + (3 + 5 + 7) + (9 + 11 + 13 + 15 + 17) + \ldots$

28. The 1st term of an A.P. is 1 and the number of terms is even. If the sum of the odd terms is 280 and of the even terms 310, find the series.

29. The interior angles of a polygon are in A.P. If the least angle is 120° and the common difference is 5°, find the number of sides.

30. Since the sum of the first n odd number is n^2, show that every odd number is the difference of two squares.

31. If the sum of p terms of an A.P. is q and the sum of q terms is p, find the sum of $(p + q)$ terms.

32. If the first term of an A.P. be a and the sum of the first p terms be zero, show that the sum of the next q terms will be $-aq(p + q) \div (p - 1)$.

33. The distances which a man travels in successive days are in A.P. At the end of the 5th day he has gone 100 miles and at the end of the 7th 150 miles. How long will he take to complete 300 miles in all?

34. How many terms of the series $15 + 21 + 27 + \ldots$ must be taken to yield the sum 231? Construct the related problem to which the negative result taken positively is an answer.

35. If a, b, c are in A.P., show the equation $(b-c)x^2+(c-a)x+a-b=0$ has equal roots.

36. If a^2, b^2, c^2 are in A.P., show that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

37. If the sum of p terms of an A.P. is equal to the sum of q terms, show that the sum of $(p + q)$ terms is 0.

CHAPTER III

GEOMETRIC PROGRESSION

19. Geometric Progression. A series in which each term, after the first, is formed by multiplying the preceding term by a constant quantity is called a **geometric progression (G.P.).**

The constant quantity is called the **common ratio** of the series and is found by dividing any term into the term which *follows* it.

The following are examples of geometric series :

(1) 1, 2, 4, 8, . . . (2) 6, 3, $\frac{3}{2}$, $\frac{3}{4}$, . . .

(3) 2, − 6, 18, − 54, . . . (4) a, ar, ar^2, ar^3 . . .

The common ratio in (1) is 2, in (2) is $\frac{1}{2}$, in (3) is − 3 and in (4) is r.

20. General Form of a G.P. If we let a be the 1st term and r the common ratio, the general form of a G.P. is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots$$

Since a and r may have any values, this series represents every geometric series.

The 4th term $= ar^3$,
the 7th term $= ar^6$,
the 20th term $= ar^{19}$.

That is, the index of r in each term is one less than the number of the term.

$\therefore$ the nth term $= ar^{n-1}$.

If the series contains n terms, and if we represent the last term by l, we have the formula

$$l = ar^{n-1}.$$

Any term of a G.P. may be found by substituting in this formula the particular values of a, r and n.

Ex. 1.—Find the 4th, 7th and nth terms of

(1) 2, 6, 18, . . . (2) 16, − 8, 4, . . .

In (1) $a = 2, r = 3$, $\therefore$ the 4th term $= 2 \cdot 3^3 = 54$,
the 7th term $= 2 \cdot 3^6 = 1458$,
the nth term $= 2 \cdot 3^{n-1}$.

In (2) $a = 16, r = -\frac{1}{2}$, $\therefore$ the 4th term $= 16(-\frac{1}{2})^3 = -2$,
the 7th term $= 16(-\frac{1}{2})^6 = \frac{1}{4}$,
the nth term $= 16(-\frac{1}{2})^{n-1}$,
$= 2^4 \frac{(-1)^{n-1}}{2^{n-1}} = \frac{(-1)^{n-1}}{2^{n-5}}$.

The nth term in (2) is ambiguous in sign, being positive when n is odd and negative when n is even.

Ex. 2.—Which term of 243, 81, 27, . . . is $\frac{1}{81}$?

Suppose $\frac{1}{81}$ is the nth term, then since $a = 243$, $r = \frac{1}{3}$,

$\therefore$ $243(\frac{1}{3})^{n-1} = \frac{1}{81}$,
$\therefore$ $3^{n-1} = 243 \times 81 = 3^5 \times 3^4 = 3^9$,
$\therefore$ $n - 1 = 9$ or $n = 10$.
$\therefore$ the 10th term is $\frac{1}{81}$. Verify this.

21. If any two terms of a G.P. are given, the 1st term and the common ratio may be found and the series will be completely determined.

Ex.—Determine the G.P. in which the 6th term is 4 and the 9th term is 32.

Let the series be $a, ar, ar^2, \ldots$
Then $ar^5 = 4$ and $ar^8 = 32$.
$\therefore$ by division $r^3 = 8$,
$\therefore$ $r = 2$,
$\therefore$ $a = \frac{1}{8}$.

The series therefore is $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$
Verify by finding the 6th and 9th terms.

22. Geometric Mean. When three quantities are in G.P., the middle one is called the **geometric mean** between the other two.

Thus, 8 is the geometric mean between 4 and 16 since 4, 8, 16 are in G.P.

If x is the geometric mean between a and b, then a, x, b are in G.P.

$$\therefore \quad \frac{x}{a} = \frac{b}{x}.$$

$$\therefore \quad x = \pm \sqrt{ab}.$$

Therefore *the geometric mean between any two quantities is the square root of their product.* It is usual to consider the positive square root only.

23. When any number of terms form a G.P. the terms intermediate between any two are called the **geometric means** between those two terms.

Thus, 3, 9, 27 are three geometric means between 1 and 81 since 1, 3, 9, 27, 81 are in G.P.

Ex.—Insert 6 geometric means between 8 and $\frac{1}{16}$.

The whole series consists of 8 terms of which the first is 8 and the 8th is $\frac{1}{16}$.

$$\therefore \quad \tfrac{1}{16} = 8r^7,$$

$$\therefore \quad r^7 = \tfrac{1}{128} = \frac{1}{2^7},$$

$$\therefore \quad r = \tfrac{1}{2}.$$

$\therefore$ the means are $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

To insert n geometric means between a and b. The whole series consists of $n + 2$ terms of which the first is a and the last or $(n + 2)$th is b.

Therefore, if r is the common ratio,

$$ar^{n+1} = b,$$

$$\therefore \quad r = \sqrt[n+1]{\frac{b}{a}} \text{ or } \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Then the required means are ar, ar^2, $ar^3 \ldots ar^n$, where

$$r = \sqrt[n+1]{\frac{b}{a}}.$$

EXERCISE 8

Find the required term in each of the following:

1. 10th term of 1, 2, 4, . . .

2. 7th term of 9, 3, 1, . . .

3. 6th term of 9, 6, 4, . . .

4. 8th term of 40, − 20, 10, . . .

5. 20th term of 1, x, x^2, . . .

6. 12th term of 32, − 16, 8, . . .

7. 11th term of a^2, ab, b^2, . . .

8. 21st term of a, $-a^2b$, a^3b^2, . . .

9. Find the 7th and nth terms of x^2, x, 1, . . .

10. Find the $(2n)$th and the $(2n + 1)$th terms of 1, $-\frac{1}{2}$, $\frac{1}{4}$, . . .

11. Find the nth term of $(a - b)^2$, $a^2 - b^2$, $(a + b)^2$, . . .

12. If 18 is the 1st term of a G.P. and $\frac{2}{3}$ is the 4th term, find the common ratio.

13. The 3rd term of a G.P. is 40 and the 5th term is 160, find the series.

14. The 3rd term of a G.P. is 1 and the 6th term is $-\frac{1}{8}$, find the 10th term.

15. The 4th term of a G.P. is ·016 and the 7th term is ·000128, find the 1st term.

16. If a, b, c, d are in G.P., show that $ad = bc$.

17. Which term of the series 5, 10, 20, . . . is 640? Which term of 54, 36, 24, . . . is $7\frac{1}{9}$?

18. What is the geometric mean between 4 and 9; 3 and 243; 16 and $\frac{1}{4}$; $(a - b)^2$ and $(a + b)^2$?

19. Insert two numbers between 5 and 135 so that the four numbers may form a G.P.

20. Insert 4 geometric means between 2 and 64; 5 between 3 and $7\frac{1}{8}$; 3 between x^4 and x^4y^4.

21. Find the geometric mean between the 5th and 9th terms of 3, 6, 12, . . .

22. The sum of the 3rd and 4th terms of a G.P. is 40 and of the 6th and 7th is 2560. Find the series.

23. Prove that the product of any two terms equidistant from the beginning and the end of a G.P. is constant.

24. If the arithmetic mean between a and b is equal to the geometric mean, show that $a = b$.

25. The geometric mean between two numbers is 8 and the arithmetic mean is 10; find the numbers.

26. The geometric mean between two numbers is greater than one of them by 20 and less than the other by 100; find the numbers.

27. If a, b, c be in G.P. and x is the geometric mean between a and b and y between b and c, show that a, x, b, y, c are in G.P.

28. Show that the reciprocals of the terms of a G.P. are in G.P. Is this true concerning an A.P. ?

29. Show that the series of rth terms of a G.P. are in G.P.

30. If all the terms of a G.P. be multiplied (or divided) by the same quantity, the results are in G.P.

31. The sum of three numbers in G.P. is 21 and their product is 216. Find the numbers.

32. Find three numbers in G.P. whose sum is 26 and the sum of their reciprocals is $1\frac{3}{8}$.

33. If a, b, c, d be in G.P., prove that

$$(a - d)^2 = (b - c)^2 + (c - a)^2 + (d - b)^2.$$

34. If $(a^2 + b^2)(b^2 + c^2) = (ab + bc)^2$, then a, b, c are in G.P.

35. If 27, x, y, 8 are in G.P., find x and y.

24. Sum of a G.P. Let the first term be a, the common ratio r, and the number of terms n.

Let S be the sum of n terms,

then $S = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$,

$\therefore Sr = \quad ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n$.

By subtracting, $S - Sr = a - ar^n$,

$\therefore S(1 - r) = a(1 - r^n)$,

$\therefore S = \dfrac{a(1 - r^n)}{1 - r}$ or $\dfrac{a(r^n - 1)}{r - 1}$.

The sum of any G.P. may be found by substituting the particular values of a, r and n in this formula. If r is less than unity it is more convenient to use the first form and the other when r is greater than unity.

Since $l = ar^{n-1}$ and therefore $lr = ar^n$,

the formula $S = \dfrac{a - ar^n}{1 - r}$ may be written $S = \dfrac{a - lr}{1 - r}$.

Ex. 1.—Find the sum of 10 terms of $2 + 4 + 8 + \ldots$

Here $a = 2, r = 2, n = 10$,

$\therefore S = \dfrac{a(r^n - 1)}{r - 1} = \dfrac{2(2^{10} - 1)}{2 - 1} = 2046.$

Ex. 2.—Sum the series $27 - 9 + 3 - 1 + \ldots$ to 7 terms.

Here $a = 27, r = -\frac{1}{3}, n = 7$,

$\therefore S = \dfrac{a(1 - r^n)}{1 - r} = \dfrac{27\{1 - (-\frac{1}{3})^7\}}{1 + \frac{1}{3}} = \dfrac{27(1 + 3^{-7})}{\frac{4}{3}} = 20\frac{7}{27}.$

Ex. 3.—In a G.P. the first term is 7, the last is 448 and the sum is 889. Find the common ratio.

Since a, l and S are given it is better to use the formula, $S = \dfrac{lr - a}{r - 1}$.

$\therefore 889 = \dfrac{448r - 7}{r - 1}, \quad \therefore r = 2.$

EXERCISE 9

Sum the following geometric series:

1. $1 + 2 + 4 + \ldots$ to 9 terms.

2. $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ to 10 terms.

3. $\frac{1}{2}\frac{1}{4} + \frac{1}{1}\frac{1}{2} + \frac{1}{6} + \ldots$ to 8 terms.

4. $16 - 8 + 4 - \ldots$ to 9 terms.

5. $1 + \sqrt{3} + 3 + \ldots$ to 8 terms.

6. $\cdot 1 + \cdot 01 + \cdot 001 + \ldots$ to 6 terms.

7. $1 - 2 + 4 - \ldots$ to $2n$ terms.

8. $1 - \frac{1}{2} + \frac{1}{4} - \ldots$ to $(2n + 1)$ terms.

9. Sum the series $a + 2a^2 + 4a^3 + \ldots$ to n terms.

10. Find the sum of $1 + 1{\cdot}04 + 1{\cdot}04^2 + 1{\cdot}04^3 + \ldots$ to 15 terms, being given that $1{\cdot}04^{\cdot 5} = 1{\cdot}80094$.

11. Find the sum of the successive powers of 2 from 2 to 4096 inclusive, if $2^{12} = 4096$.

12. The sum of the first 10 terms of a G.P. is 33 times the sum of the first 5 terms. Find the common ratio.

13. Show that there are two geometric series in which the 1st term is 18 and the 3rd term is 8 and find the sum of each to 5 terms.

14. If n be an even integer, prove that

$$(1 + x + x^2 + \ldots + x^n)(1 - x + x^2 - \ldots + x^n) = 1 + x^2 + x^4 + \ldots + x^{2n}.$$

15. The sum of a G.P. whose common ratio is 3 is 728 and the last term is 486; find the first term.

16. Find the sum of $2n$ terms of a series in which every even term is a times the term before it and every odd term is b times the term before it, the first term being 1.

17. If a, b, c, d be in G.P. prove that $a + b, b + c, c + d$ will be in G.P. and also $a^2 + b^2, b^2 + c^2, c^2 + d^2$.

18. If xy, y^2, z^2 are in A.P., then $y, z, 2y - x$ are in G.P.

19. If x, y, z be the pth, qth, rth terms respectively of a G.P., prove that $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = 1$.

20. If S_n, S_{2n}, S_{3n} be the sum of n, $2n$, $3n$ terms of a G.P., show that $S_n(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$.

21. If S be the sum of an odd number of terms in G.P. and S_1 be the sum when the signs of the even terms are changed, show that the sum of the squares of the terms is SS_1.

25. Limiting Values. Consider the quantity r^n, where r has a certain fixed value while n may be any positive integer.

(1) It is clear that if r is positive and less than unity, as n increases r^n decreases.

Thus if $r = \frac{1}{2}$,

when $n = 2$, $r^n = \frac{1}{4} = \cdot 25$,

when $n = 3$, $r^n = \frac{1}{8} = \cdot 125$,

when $n = 10$, $r^n = \frac{1}{1024} = \cdot 000976 \ldots$,

when $n = 20$, $r^n = \frac{1}{1024^2} = \cdot 0000009 \ldots$, and so on.

Also when $r < 1$, we can make r^n as small as we please by taking n sufficiently large. That is, as n becomes infinitely great r^n approaches the value 0.

The symbol ∞ is used to represent a quantity which is infinitely great or infinity, and the preceding statement is written in brief form thus :

If $r < 1$, when $n = \infty$, the *limit* of r^n is 0.

(2) If $r > 1$, as n increases r^n also increases, and we can make r^n as large as we please by taking n sufficiently large. *There is therefore no limit to the value of* r^n *when* $r > 1$.

26. The Infinite Geometric Progression. If the number of terms of a geometric progression is unlimited it is called an **infinite G.P.**

Suppose it is required to find the sum of the unlimited series, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

We might find the sum graphically as follows:

Let OA be a line 2 units in length. Bisect OA at B, BA at C, CA at D, and so on without limit.

Then $OB = 1$, $BC = \frac{1}{2}$, $CD = \frac{1}{4}$, etc.

Therefore the sum of the given series is equal to the sum of OB, BC, CD, etc.

It is evident as the number of bisections is continued, the last division approaches closer to the point A, and may be made to approach as closely as we wish by making the number of division sufficiently large.

Therefore the limit of the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ is 2, when the number of terms is unlimited.

In the series $1 + 2 + 4 + 8 + \ldots$, the common ratio is greater than 1 and therefore the terms are increasing. It is evident that in an increasing G.P. the sum may be made as large as we please by making the number of terms sufficiently large.

But in the series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$, the common ratio is less than 1, and the terms are continually decreasing. In a decreasing G.P. the sum of an unlimited number of terms is limited as we will show by examining the sum of the general series.

The formula for the sum of n terms of $a + ar + ar^2 \ldots$ is

$$S = \frac{a(1 - r^n)}{1 - r},$$

This may be written $S = \dfrac{a}{1 - r} - \dfrac{ar^n}{1 - r}$.

Now if $r < 1$ and n is indefinitely large r^n is indefinitely small and therefore $\frac{ar^n}{1-r}$ is also indefinitely small.

Therefore *when* $r < 1$, *the limit of the sum of the infinite series* $a + ar + ar^2 + \ldots$ is $\frac{a}{1-r}$,

or $a + ar + ar^2 + \ldots$ to $\infty = \frac{a}{1-r}$.

Ex. 1.—Sum $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ to infinity.

Since $a = 1$ and $r = \frac{1}{2}$, $S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

Ex. 2.—Find the limit of the sum of $2 - \frac{2}{3} + \frac{2}{9} - \ldots$

Since $a = 2$ and $r = -\frac{1}{3}$, $S = \frac{2}{1+\frac{1}{3}} = \frac{3}{2}$.

EXERCISE 10

Find the limit of the sum of :

1. $4 + 2 + 1 + \ldots$
2. $1 + \frac{1}{3} + \frac{1}{9} + \ldots$
3. $1 - \frac{1}{2} + \frac{1}{4} - \ldots$
4. $5 + \frac{10}{3} + \frac{20}{9} + \ldots$
5. $3 + \sqrt{3} + 1 + \ldots$
6. $\cdot5 + \cdot05 + \cdot005 + \ldots$
7. $a - \frac{1}{4}a + \frac{1}{16}a - \ldots$
8. $5 + \sqrt{15} + 3 + \ldots$
9. $1 + \frac{1}{1\cdot05} + \frac{1}{1\cdot05^2} + \ldots$
10. Sum to infinity $1 + \frac{1}{1\cdot025} + \frac{1}{1\cdot025^2} + \ldots$
11. When $r < 1$, find the sum to infinity of $a - ar + ar^2 - \ldots$
12. In an infinite G.P., $S = 9$ and $a = 6$, find r.
13. Since $\cdot\dot{4} = \cdot4 + \cdot04 + \cdot004 + \ldots$ find a vulgar fraction which is the limit of $\cdot\dot{4}$.

14. What fraction is $\cdot 3\dot{5}\dot{4}$ or $\cdot 3 + \cdot 054 + \cdot 00054 + \ldots$ equal to?

15. Reduce $\cdot\dot{6}$, $\cdot\dot{1}\dot{3}$, $\cdot 01\dot{4}$ to equivalent fractions.

16. Show that any term of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ is equal to the sum of all the terms which follow it.

17. Construct a series whose 1st term is 1 and in which any term is equal to two-thirds of the sum of all the terms which follow it.

18. The sum of the first two terms of a G.P. is 5 and every term is three times the sum of all that follow it; find the series.

19. Show that $\dfrac{1+x^2}{(1-x)^2} = 1 + \dfrac{2x}{1+x^2} + \dfrac{4x^2}{(1+x^2)^2} + \ldots$ to infinity.

20. By how much does the sum of 7 terms of $1 + \frac{1}{3} + \frac{1}{9} + \ldots$ differ from the sum to infinity?

21. Show that the sum of $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ to infinity differs from the sum to 12 terms by less than $\cdot 001$.

22. Find the G.P. whose 2nd term is -8 and whose sum to infinity is 18.

23. Find the G.P. of which the sum of the first two terms is $2\frac{2}{3}$ and the sum to infinity is $4\frac{1}{5}$.

24. Prove that the sums of the two series $1 + \frac{3}{5} + \frac{9}{25} + \ldots$ and $\frac{7}{4} + \frac{7}{8} + \frac{7}{16} + \ldots$ approach each other without limit as the number of terms is indefinitely increased.

25. What assumption must be made if it is possible to sum

$$2 + \frac{4x}{5} + \frac{8x^2}{25} + \ldots \text{ to infinity?}$$

26. A ball falls to the ground from a height of 32 feet and always rises $\frac{3}{4}$ of the height from which it fell. Find the whole distance through which it will move before coming to rest.

27. If $x = 1 + a + a^2 + \ldots$ to infinity, and $y = 1 + b + b^2 + \ldots$ to infinity, prove that $1 + ab + a^2b^2 + \ldots$ to infinity $= \dfrac{xy}{x + y - 1}$.

28. A person is entitled to an annual payment which for each year is 10% less than the year before. Show that however long he may live, he cannot receive more than a certain sum in all.

29. If $P = 1 + r^p + r^{2p} + \ldots$ to inf., and $Q = 1 + r^q + r^{2q} + \ldots$ to inf., show that $P^q(Q - 1)^p = Q^p(P - 1)^q$.

30. An air pump, at each stroke of the piston, removes $\frac{1}{2}$ of the air in the vessel. Show that less than 1% of the air will remain after seven strokes.

31. The middle points of the sides of a triangle are joined, the middle points of the sides of the triangle so formed are joined, and so on without limit. Show that the sum of the areas of all the triangles so formed is one-third of the area of the original triangle.

32. Find an infinite geometric series whose sum is $\frac{2}{3 - 2x}$.

33. If S_n is the sum of the nth powers of the terms of an infinite G.P., show that $\frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \ldots$ to inf. $= \frac{1}{a - 1} - \frac{r}{a - r}$.

34. If S_n denote the sum of n terms of $a + ar + ar^2 + \ldots$, show that $S_1 + S_2 + S_3 + \ldots + S_n = \frac{a}{r - 1}\left(r \cdot \frac{r^n - 1}{r - 1} - n\right)$.

35. If P be the product of n terms in G.P., S their sum, and S_1 the sum of their reciprocals, then $P^2 = \left(\frac{S}{S_1}\right)^n$.

36. If S be the sum and S_1 the sum of the squares of the terms of an infinite G.P., show that the sum to n terms is

$$S\left\{1 - \left(\frac{S^2 - S_1}{S^2 + S_1}\right)^n\right\}.$$

37. From three numbers in G.P. three others in G.P. are subtracted. If the remainders are in G.P., then all three series have the same common ratio.

CHAPTER IV

HARMONIC PROGRESSION

27. Harmonic Progression. A sequence of quantities is said to be in **harmonic progression** when their reciprocals are in arithmetic progression.

Thus, $1, \frac{1}{2}, \frac{1}{3}, \ldots$ are in H.P. since $1, 2, 3, \ldots$ are in A.P.
Also $15, 12, 10, \ldots$ are in H.P. since $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}, \ldots$ are in A.P.

28. General Form of an H.P. Since the general form of an A.P. is

$$a, a + d, a + 2d, \ldots, a + (n - 1)d,$$

the general form of an H.P. is

$$\frac{1}{a}, \frac{1}{a + d}, \frac{1}{a + 2d}, \ldots, \frac{1}{a + (n - 1)d}.$$

Any term of an H.P. may be found by finding the same term of the corresponding A.P. and taking its reciprocal.

There is no general formula for the sum of an H.P.

Ex. 1.—Find the 6th term and nth term of an H.P. whose first two terms are 20 and 15.

The first two terms of the corresponding A.P. are $\frac{1}{20}, \frac{1}{15}$,
$\therefore$ the common difference of the A.P. $= \frac{1}{15} - \frac{1}{20} = \frac{1}{60}$,
$\therefore$ the 6th term of the A.P. $= \frac{1}{20} + \frac{5}{60} = \frac{2}{15}$,
and the nth term of the A.P. $= \frac{1}{20} + (n - 1)\frac{1}{60} = \frac{n+2}{60}$,
$\therefore$ the 6th term of the H.P. $= \frac{15}{2}$ or $7\frac{1}{2}$,
and the nth term of the H.P. $= \frac{60}{n+2}$.

Ex. 2.—The 12th term of an H.P. is $\frac{1}{5}$ and the 19th is $\frac{3}{22}$, find the 4th term.

Let a be the 1st term and d the common difference of the corresponding A.P.

Then $a + 11d = 5$ and $a + 18d = \frac{22}{3}$.

Solving we find $a = \frac{4}{3}$ and $d = \frac{1}{3}$.

$\therefore$ the 4th term of the A.P. $= a + 3d = \frac{7}{3}$,

$\therefore$ the 4th term of the H.P. $= \frac{3}{7}$.

29. Harmonic Mean. When three terms are in H.P. the middle one is called the **harmonic mean** between the other two.

Let x be the harmonic mean between a and b. Then $\frac{1}{x}$ is the arithmetic mean between $\frac{1}{a}$ and $\frac{1}{b}$,

$$\therefore \quad \frac{1}{x} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{a+b}{2ab},$$

$$\therefore \quad x = \frac{2ab}{a+b}.$$

Therefore *the harmonic mean between two quantities is twice their product divided by their sum.*

When three numbers are given in H.P., they may therefore be conveniently represented by $a, \frac{2ab}{a+b}, b$.

Ex. 1.—Insert three harmonic means between 2 and 8.

The 1st term of the corresponding A.P. is $\frac{1}{2}$ and the 5th is $\frac{1}{8}$,

$$\therefore \quad \tfrac{1}{8} = \tfrac{1}{2} + 4d, \qquad \therefore \quad d = -\tfrac{3}{32}.$$

$\therefore$ the terms of the arithmetic series are $\frac{1}{2}, \frac{13}{32}, \frac{10}{32}, \frac{7}{32}, \frac{1}{8}$.

$\therefore$ the required harmonic means are $\frac{32}{13}, \frac{16}{5}, \frac{32}{7}$.

Ex. 2.—If a, b, c are in H.P., prove $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{c} + \frac{1}{a}$.

Since a, b, c are in H.P., $b = \frac{2ac}{a+c}$,

$$\therefore \frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{\frac{2ac}{a+c} - a} + \frac{1}{\frac{2ac}{a+c} - c} = \frac{a+c}{ac-a^2} + \frac{a+c}{ac-c^2}$$

$$= \frac{a+c}{a(c-a)} + \frac{a+c}{c(a-c)} = \frac{c(a+c) - a(a+c)}{ac(c-a)} = \frac{c^2-a^2}{ac(c-a)} = \frac{c+a}{ac}$$

$$= \frac{1}{a} + \frac{1}{c}.$$

30. If a, b, c are in H.P., then $\frac{a}{c} = \frac{a-b}{b-c}$.

Since a, b, c are in H.P., $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b},$$

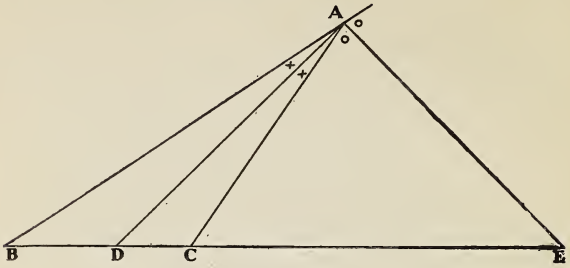
$$\therefore \frac{a-b}{ab} = \frac{b-c}{bc}, \text{ or } \frac{a-b}{a} = \frac{b-c}{c},$$

$$\therefore \frac{a}{c} = \frac{a-b}{b-c}.$$

Therefore, if three numbers are in H.P., the first is to the third as the difference between the first and second is to the difference between the second and third.

This property of three numbers in H.P. is sometimes given as the definition of an H.P. of three terms.

The figure shows three lines whose lengths are in H.P.



The interior and exterior vertical angles of the triangle ABC are bisected by AD and AE which meet the base at D and E.

Then BE, BC and BD are in H.P.

It is proven in geometry that $\frac{BA}{AC} = \frac{BD}{DC} = \frac{BE}{EC}$

$$\therefore \quad \frac{BE}{BD} = \frac{EC}{DC}, \qquad \therefore \quad \frac{BE}{BD} = \frac{BE - BC}{BC - BD}$$

$$\therefore \qquad BE,\ BC,\ BD \text{ are in H.P.}$$

Ex.—Three numbers are in H.P. What must be subtracted from each so that the results will be in G.P.?

Let the numbers be a, $\frac{2ab}{a+b}$, b and let x be subtracted from each.

Then $\qquad a - x,\ \frac{2ab}{a+b} - x,\ b - x$ are in G.P.

$$\therefore \quad (a-x)(b-x) = \left(\frac{2ab}{a+b} - x\right)^2, \qquad \therefore \quad x = \frac{ab}{a+b}.$$

Therefore the quantity to be subtracted is half of the middle term. Verify the correctness of this.

EXERCISE 11

Show that the following are in harmonic progression and find the next two terms in each :

1. 60, 30, 20, . . . **2.** $\frac{3}{2}$, 1, $\frac{3}{4}$, . . . **3.** 4, 2, $\frac{4}{3}$, . . .

4. 4·8, 6, 8, . . . **5.** 6, − 12, − 3, . . .

6. $\frac{1}{\sqrt{2}}$, $\sqrt{2} - 1$, $\frac{2 - \sqrt{2}}{2}$, . . .

7. Find the 24th term of the H.P., 24, 12, 8, . . .

8. Find the nth term of the H.P., $\frac{1}{2}$, $\frac{4}{7}$, $\frac{2}{3}$, . . .

9. Find the nth term of an H.P. whose 1st term is a and 2nd term is b.

10. The 5th term of an H.P. is 168 and the 8th is 105, find the first two terms.

11. The 1st term of an H.P. is 2·4 and the 4th term is − 3, find which term is − ·3.

12. What is the harmonic mean between 30 and 15, − 3 and 4; $\frac{1}{7}$ and $\frac{1}{5}$; $\frac{a - b}{a + b}$ and $\frac{a + b}{a - b}$?

13. Insert three harmonic means between 1 and 2; 4 between − 7 and $\frac{1}{2}$; 10 between 3 and $\frac{1}{4}$.

14. If a, x, y, b be in H.P., find x and y in terms of a and b.

15. Insert n harmonic means between a and b.

16. Find the sum of 6 terms in H.P. if the 1st term is 60 and the 2nd is 30.

17. If a, b, c are in H.P., prove that $\frac{a}{a - b} = \frac{a + c}{a - c}$.

18. If a, b, c are in H.P., prove that $\frac{b + a}{b - a} + \frac{b + c}{b - c} = 2$.

19. Find two numbers whose difference is 8 and the harmonic mean between them is 15.

20. The sum of three numbers in H.P. is 37. If the middle one is 12, find the others.

21. Find two numbers whose arithmetic mean is 25 and whose harmonic mean is 24.

22. The sum of three terms in H.P. is $1\frac{1}{12}$ and the 1st term is $\frac{1}{2}$. Find the series and continue it one term each way.

23. What number must be added to each of the numbers 1, 3, 9 in order that the sums may be in H.P.?

24. Find x so that $a + x$, $b + x$, $c + x$ may be in H.P.

25. When n is finite can the nth term of an A.P. be infinity? Can the nth term of an H.P. be infinity? Can it be 0?

26. Insert 9 harmonic means between 1 and -1.

27. If the nth term of an H.P. is m and the mth term is n, prove that the rth term is $\frac{mn}{r}$.

31. If A, G, H are the arithmetic, geometric and harmonic means respectively between any two positive quantities, then A, G and H are in G.P., and they are in descending order of magnitude.

Let the quantities be x and y.

Then $$A = \frac{x+y}{2},\ G = \sqrt{xy},\ H = \frac{2xy}{x+y},$$

$$\therefore\ AH = \frac{x+y}{2} \times \frac{2xy}{x+y} = xy = G^2,$$

$\therefore$ A, G, H are in G.P. and therefore G lies in magnitude between A and H.

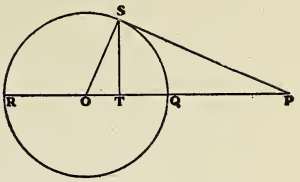
$$A - G = \frac{x+y}{2} - \sqrt{xy} = \frac{x - 2\sqrt{xy} + y}{2} = \frac{(\sqrt{x} - \sqrt{y})^2}{2}.$$

Since x and y are positive and therefore $\sqrt{x}$ and $\sqrt{y}$ are real quantities, it follows that $(\sqrt{x} - \sqrt{y})^2$ is a positive quantity.

$\therefore$ $A - G$ is positive and therefore $A > G$, and since G lies between A and H in magnitude, $G > H$.

$$\therefore\ A > G > H.$$

In the figure, PS is a tangent to the circle whose centre is O. The lengths of PO, PS and PT are respectively the A, G and H means between the lengths of PQ and PR.



For $PQ + PR = 2PQ + 2QO = 2PO$, $\therefore$ PO is the A mean, and $PS^2 = PQ \,.\, PR$, $\therefore$ PS is the G mean

and
$$\frac{2PQ \,.\, PR}{PQ + PR} = \frac{2PS^2}{2PO} = \frac{PO \,.\, PT}{PO} = PT,$$

$\therefore$ PT is the H mean.

It is easily shown that

$$PO > PS \text{ and } PS > PT,$$
$$\therefore \quad A > G > H.$$

32. Sometimes a series which is to be summed is neither arithmetic nor geometric, but is the sum of two or more series each of which is arithmetic or geometric.

Ex. **1.**—Sum to n terms the series

$$(a + b) + (2a + b^2) + (3a + b^3) + (4a + b^4) + \ldots$$

This series is the sum of the A.P., $a + 2a + 3a + \ldots$ and the G.P., $b + b^2 + b^3 + \ldots$

$\therefore$ $S = (a + 2a + 3a + \ldots$ to n terms$) + (b + b^2 + b^3 + \ldots$ to n terms$)$

$$= \frac{n}{2}\left\{2a + (n-1)a\right\} + b \,.\, \frac{b^n - 1}{b - 1}$$

$$= \frac{an(n+1)}{2} + b \,.\, \frac{b^n - 1}{b - 1}.$$

Ex. **2.**—Sum to n terms the series

$$1 + 2\tfrac{1}{2} + 3\tfrac{1}{4} + 4\tfrac{1}{8} + \ldots$$

$S = (1 + 2 + 3 + \ldots$ to n terms$) + (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ to $n - 1$ terms$)$

$$= \frac{n(n+1)}{2} + \frac{1}{2} \,.\, \frac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}} = \frac{n(n+1)}{2} + 1 - \frac{1}{2^{n-1}}.$$

Ex. 3.—The nth term of a series is $3^n + 2^n$, find its sum to n terms.

If the nth term is $3^n - 2^n$, then the 1st term is $3^1 - 2^1$, the 2nd is $3^2 - 2^2$, and so on.

$$\therefore \quad S = (3 - 2) + (3^2 - 2^2) + (3^3 - 2^3) + \ldots \text{ to } n \text{ terms.}$$
$$= (3 + 3^2 + 3^3 + \ldots \text{ to } n \text{ terms}) - (2 + 2^2 + 2^3 + \ldots \text{ to } n \text{ terms})$$
$$= \tfrac{3}{2}(3^n - 1) - 2(2^n - 1) = \tfrac{1}{2} \cdot 3^{n+1} - 2^{n+1} + \tfrac{1}{2}.$$

Ex. 4.—Sum to n terms (1) $9 + 99 + 999 + \ldots$
(2) $4 + 44 + 444 + \ldots$

(1) $S = (10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots$ to n terms.
$= (10 + 10^2 + 10^3 + \ldots$ to n terms$) - (1 + 1 + 1 \ldots$ to n terms$)$
$= \tfrac{10}{9}(10^n - 1) - n.$

(2) $S = 4 + 44 + 444 + \ldots = \tfrac{4}{9}(9 + 99 + 999 + \ldots)$
$= \tfrac{4}{9}[\tfrac{10}{9}(10^n - 1) - n] = \tfrac{40}{81}(10^n - 1) - \tfrac{4}{9}n.$

Check by letting $n = 3$ in each case.

EXERCISE 12

1. If the arithmetic mean between two numbers is 3 and the harmonic mean is $2\frac{2}{5}$, find the numbers.

2. The arithmetic mean between two numbers is 30 and the geometric mean is 24; find the harmonic mean.

3. If the arithmetic mean between two numbers is 1, show that the harmonic mean is the square of the geometric mean.

4. To each of three consecutive terms of a G.P. the second term of the three is added. Show that the resulting quantities are in H.P.

5. If the harmonic mean between two numbers is $\frac{4}{5}$ of the geometric mean, show that one number is four times the other.

6. The sum of four numbers in G.P. is 60 and the arithmetic mean between the first and last is 18. Find the numbers.

7. If x, y, z are in A.P. and x, my, z are in G.P., prove that x, m^2y, z are in H.P.

8. Show that a, b, c are in A.P., G.P., or H.P. according as

$$\frac{a-b}{b-c} = \frac{a}{a}, \frac{a}{b} \text{ or } \frac{a}{c} \text{ respectively.}$$

9. If a, b, c, d, e are such that a, b, c are in A.P., b, c, d in G.P., c, d, e in H.P., prove that a, c, e are in G.P.

10. If a, b, c are in H.P., prove $\frac{1}{a-b} + \frac{1}{b-c} + \frac{4}{c-a} = \frac{1}{c} - \frac{1}{a}$.

11. If $\frac{a}{b} = \frac{c}{d}$ and a, b, c are in A.P., then b, c, d are in H.P.

12. If $b + c, c + a, a + b$ are in H.P., then a^2, b^2, c^2 are in A.P.

13. Prove that the nth, pth and qth terms of a G.P. are in G.P. if n, p and q are in A.P.

14. If x, y, z are in G.P., $y, z, 4$ in H.P. and $\frac{1}{4}, x, y$ in A.P., find x, y and z.

15. If a is one of the 19 arithmetic means inserted between 2 and 3 and h is the corresponding harmonic mean, show that

$$a = 5 - \frac{6}{h}.$$

16. Express $a + b + \frac{a}{2} + \frac{2b}{3} + \frac{a}{4} + \frac{4b}{9} + \ldots$ as the sum of two G.P.'s, and thus find its sum to 20 terms and to infinity.

17. Sum $a - br + ar^2 - br^3 + ar^4 - br^5 + \ldots$ to $2n$ terms.

18. Sum to n terms (1) $1{\cdot}1 + 2{\cdot}01 + 3{\cdot}001 + \ldots$

(2) $\cdot 9 + \cdot 99 + \cdot 999 + \ldots$

19. Sum to n terms (1) $1\frac{1}{2} + 2\frac{1}{4} + 3\frac{1}{8} + \ldots$

(2) $1 + 3 + 7 + 15 + \ldots + (2^r - 1) + \ldots$

20. Sum $\left(r - \frac{1}{r}\right)^2 + \left(r^2 - \frac{1}{r^2}\right)^2 + \left(r^3 - \frac{1}{r^3}\right)^2 + \ldots$ to n terms.

21. If H is the harmonic mean between a and b, show that it is also the harmonic mean between $H - a$ and $H - b$.

22. There are two series, one in G.P. and one in H.P. The first two terms of each are a and b. Show that b^3 will be the product of the 3rd terms if $2a - b = 1$.

23. Show that if $\frac{a}{1+ak}$, $\frac{b}{1+bk}$, $\frac{c}{1+ck}$ are in H.P. for one value of k, they are so for all values of k.

24. If A, G, H are the arithmetic, geometric and harmonic means respectively between any two numbers, prove that

$$\frac{1}{A+G} + \frac{1}{G+H} = \frac{1}{G}.$$

25. Prove that $bc - ad$ is positive, zero or negative according as a, b, c, d are in A.P., G.P. or H.P., if a and d have like signs.

26. If x, y, z be the pth, qth and rth terms of an H.P., show that

$$xy(p-q) + yz(q-r) + zx(r-p) = 0.$$

27. If between two numbers there be inserted two arithmetic means A_1, A_2, two geometric means G_1, G_2 and two harmonic means H_1, H_2, prove that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1+A_2}{H_1+H_2}.$$

28. If $x^a = y^b = z^c$ and x, y, z are in G.P., then a, b, c will be in H.P.

29. If a, b, c are in H.P., then $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A.P., and $\frac{bc}{b+c}$, $\frac{ca}{c+a}$, $\frac{ab}{a+b}$ are in H.P.

30. The sum of three numbers in H.P. is 6·5 and the sum of their squares is 15·25; find the numbers.

CHAPTER V

RATIO AND PROPORTION

33. Ratio. The **ratio** of a to b is the quotient obtained when a is divided by b. A ratio may therefore be expressed as an ordinary fraction.

The ratio of a to b is usually printed in the form $a : b$, but in writing it, it is advisable to use the more familiar form $\frac{a}{b}$.

In the ratio $a : b$, a and b are called the **terms** of the ratio, a being called the **antecedent** and b the **consequent.**

34. Comparison of Ratios. Two ratios are equal when the fractions which represent them are equal.

Thus, $3 : 7 = 6 : 14$ because $\frac{3}{7} = \frac{6}{14}$.

If two ratios are unequal we can determine which is the greater by reducing the equivalent fractions to a common denominator.

Thus, $5 : 6$ is greater than $4 : 5$ since $\frac{5}{6}$ or $\frac{25}{30}$ is greater than $\frac{4}{5}$ or $\frac{24}{30}$.

35. Compounding Ratios. Two ratios are compounded by multiplying together the fractions which they represent.

Thus, the ratio compounded of $a : b$ and $c : d$ is $ac : bd$, since

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

When a ratio is compounded with itself the resulting ratio is called the **duplicate** of the given ratio.

Thus, the duplicate of $a : b$ is $a^2 : b^2$.

36. Proportion. When the ratio of $a : b$ is equal to the ratio of $c : d$, the four quantities a, b, c, d are said to be in **proportion** or to be **proportionals.**

Thus, 2, 5, 14, 35 are in proportion since $\frac{2}{5} = \frac{14}{35}$.

The four quantities are called the **terms** of the proportion, a and d being called the **extremes** and b and c the **means.**

When $\frac{a}{b} = \frac{c}{d}$ it follows that $ad = bc$.

Therefore *the product of the extremes is equal to the product of the means.*

When four quantities are in proportion, the 4th term is called the **fourth proportional** to the other three.

In the diagram, if DE is parallel to BC, then $AD : DB = AE : EC$,

$\therefore$ EC is the fourth proportional to AD, DB, AE.

Ex.—Find the fourth proportional to 5, 7 and 35.

Let x be the fourth proportional.

Then $\frac{5}{7} = \frac{35}{x}$, $\quad \therefore \quad x = 49.$

$\therefore$ the fourth proportiona is 49.

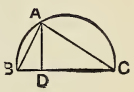
37. Continued Proportion. When $a : b = b : c$, the quantities a, b, c are said to be in **continued proportion,** and b

is called the **mean proportional** between a and c, while c is called the **third proportional** to a and b.

If $\frac{a}{b} = \frac{b}{c}$, then $b^2 = ac$ or $b = \pm\sqrt{ac}$,

therefore *the mean proportional to two numbers is the square root of their product.*

In the diagram, since $BD : DA = AD : DC$, AD is the mean proportional to BD and DC. Between what two lines is AB a mean proportional; also AC?



When $\frac{a}{b} = \frac{b}{c}$, a, b, c are in geometric progression, therefore quantities which are in continued proportion are also in G.P., and the mean proportional is the same as the geometric mean.

Ex. 1.—If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{la + mb}{pa + qb} = \frac{lc + md}{pc + qd}$.

Since $\frac{a}{b} = \frac{c}{d}$, let each $= k$,

$\therefore \quad a = bk$, and $c = dk$.

$\therefore \quad \frac{la + mb}{pa + qb} = \frac{lbk + mb}{pbk + qb} = \frac{lk + m}{pk + q}$,

and $\frac{lc + md}{pc + qd} = \frac{ldk + md}{pdk + qd} = \frac{lk + m}{pk + q}$,

$\therefore \quad \frac{la + mb}{pa + qb} = \frac{lc + md}{pc + qd}$.

Ex. 2.—If a, b, c, d are in continued proportion, prove

$$\frac{a^3 + b^3 + c^3}{b^3 + c^3 + d^3} = \frac{a}{d}.$$

Since a, b, c, d are in continued proportion and therefore in G.P., let $b = ar$, $c = ar^2$, $d = ar^3$,

$$\therefore \quad \frac{a^3 + b^3 + c^3}{b^3 + c^3 + d^3} = \frac{a^3 + a^3r^3 + a^3r^6}{a^3r^3 + a^3r^6 + a^3r^9} = \frac{a^3(1 + r^3 + r^6)}{a^3r^3(1 + r^3 + r^6)} = \frac{1}{r^3}.$$

Also
$$\frac{a}{d} = \frac{a}{ar^3} = \frac{1}{r^3},$$

$$\therefore \quad \frac{a^3 + b^3 + c^3}{b^3 + c^3 + d^3} = \frac{a}{d}.$$

EXERCISE 13

1. Find a fourth proportional to (1) 5, 6, 15. (2) $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{5}$.
(3) a, b, c. (4) $a - b$, $a^2 - b^2$, $(a - b)^2$.

2. Find a mean proportional to (1) 3, 27. (2) $\frac{1}{2}$, $\frac{1}{15}$.

(3) $\frac{a + b}{a - b}$, $\frac{a - b}{a + b}$. (4) $\sqrt{10} + 1$, $\sqrt{10} - 1$.

3. Find a third proportional to (1) 4, 12. (2) $x - y$, $x^2 - y^2$.

4. If $5 - x : x - 11 = 3 : 2$, find x and verify.

5. What must be added to each term of the ratio 3 : 7 so that the resulting ratio will be 4 : 5 ?

6. Two numbers are in the ratio 6 : 7. If three times the first exceeds twice the second by 24, find the numbers.

7. If $x : y = 2 : 5$, find the ratio of $7x - 2y$ to $6x - 2y$.

8. Divide 240 into four parts in the ratio 2 : 3 : 5 : 6.

9. If the ratio $a + x : b + x$ is the duplicate of $a : b$, find x. (The duplicate of $a : b = a^2 : b^2$.)

10. Find two numbers whose difference is 5 so that the ratio of their sum to the sum of their squares is 1 : 13.

11. Show that the ratio $a : b$ is the duplicate of $a + c : b + c$ if $ab = c^2$.

12. If $a : b = c : d$, show that

(1) $a^2 + ab : c^2 + cd = ab : cd$.

(2) $(ab + cd)^2 = (a^2 + c^2)(b^2 + d^2)$.

(3) $\frac{a}{a - b} : \frac{a + b}{b} = \frac{c}{c - d} : \frac{c + d}{d}$.

13. If a, b, c are in continued proportion, prove that

(1) $a^3 + b^3 : b^3 + c^3 = ab : c^2$.

(2) $(ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$.

(3) $\sqrt{ab} + \sqrt{bc}$ is a mean proportional to $a + b$ and $b + c$.

14. If a, b, c, d are in continued proportion, prove (1) $a : d = b^3 : c^3$. (2) $(a + b)(c + d) = (b + c)^2$. (3) $ab + bc + cd$ is a mean proportional to $a^2 + b^2 + c^2$ and $b^2 + c^2 + d^2$.

15. Show that $\sqrt{3} + 1$, $2\sqrt{3} + 4$, $6\sqrt{3} + 10$ and $16\sqrt{3} + 28$ are in continued proportion.

16. If $ab + cd$ is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$, show that $a : b = c : d$.

17. If $\dfrac{a}{y+z} = \dfrac{b}{z+x} = \dfrac{c}{x+y}$, then $\dfrac{x}{b+c-a} = \dfrac{y}{c+a-b} = \dfrac{z}{a+b-c}$.

18. If

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

prove that $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$.

19. If $a + b$, $b + c$, $c + a$ are in continued proportion, show that $a + b$, $b + c$, $c - a$, $a - b$ are four proportionals. Show that the converse is also true.

20. If a, b, c, d are proportionals, prove that $a^3c + ac^3$, $b^3d + bd^3$, a^2c^2 and b^2d^2 are proportionals. Show that the converse is not necessarily true.

38. The ratio $a : b$ is said to be a **ratio of greater inequality,** a **ratio of equality,** or a **ratio of less inequality** according as $a > = \text{ or } < b$.

Thus, $6 : 5$ or $\frac{6}{5}$ is a ratio of greater inequality,
$5 : 5$ or $\frac{5}{5}$ is a ratio of equality,
$3 : 5$ or $\frac{3}{5}$ is a ratio of less inequality.

39. A Ratio of Greater Inequality is decreased and a Ratio of Less Inequality is increased by adding the same Positive Quantity to both its Terms.

Let $\frac{a}{b}$ be the given ratio, in which b is positive and let x be the positive quantity which is added to each term.

Then $\frac{a}{b} - \frac{a+x}{b+x} = \frac{a(b+x) - b(a+x)}{b(b+x)} = \frac{x(a-b)}{b(b+x)}$.

If $a > b$, this difference is positive,

$$\therefore \quad \frac{a+x}{b+x} < \frac{a}{b}.$$

Similarly, if. $\quad a < b, \frac{a+x}{b+x} > \frac{a}{b}$.

$$\therefore \quad \frac{a+x}{b+x} < \text{ or } > \frac{a}{b} \text{ according as } a > \text{ or } < b.$$

Cor. 1. If $a = b$, the ratio is unchanged by adding the same quantity to both terms.

Cor. 2. A ratio of greater inequality is increased and a ratio of less inequality is decreased by *subtracting* the same positive quantity from both terms.

Cor. 3. Any fraction is made more nearly equal to unity by adding the same positive quantity to both terms.

40. Finding a Ratio by Solving an Equation.

Ex. 1.—If $7x - 3y = 0$, find the ratio $x : y$.

Since $\quad 7x = 3y, x = \frac{3y}{7}$ or $\frac{x}{y} = \frac{3}{7}$.

$$\therefore \quad x : y = 3 : 7.$$

It will be seen that the ratio of $x : y$ can be found from any equation of the first degree in x and y in which the absolute term is 0.

Ex. 2.—If $15x^2 + 2xy - 8y^2 = 0$, find $x : y$.

Factoring, $\quad (3x - 2y)(5x + 4y) = 0,$

$$\therefore \quad 3x - 2y = 0 \text{ or } 5x + 4y = 0,$$

$$\therefore \quad \frac{x}{y} = \frac{2}{3} \text{ or } -\frac{4}{5}.$$

This equation might be written in the form in which $\frac{x}{y}$ is the unknown by dividing by y^2, then

$$15\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right) - 8 = 0.$$

From what type of equations of the second degree in x and y can the ratio of $x : y$ be found?

Ex. 3.—Find $x : y$ if $4x^2 - 7xy - 10y^2 = 0$.

Here $$4\left(\frac{x}{y}\right)^2 - 7\left(\frac{x}{y}\right) - 10 = 0,$$

$$\therefore \quad \frac{x}{y} = \frac{7 \pm \sqrt{49 + 160}}{8} = \frac{7 \pm \sqrt{209}}{8}.$$

Ex. 4.—If $x - 2y + z = 0$, $3x + 2y - 3z = 0$, find $x : y : z$.

Eliminate y and we get $2x = z$, or $\frac{x}{z} = \frac{1}{2}$.

Eliminate z and we get $3x = 2y$, or $\frac{x}{y} = \frac{2}{3}$.

$$\therefore \quad \frac{x}{1} = \frac{z}{2} \text{ and } \frac{x}{2} = \frac{y}{3}, \qquad \therefore \quad \frac{x}{2} = \frac{y}{3} = \frac{z}{4}.$$

41. If $ax + by + cz = 0$ and $a_1x + b_1y + c_1z = 0$, prove

$$\frac{x}{bc_1 - b_1c} = \frac{y}{ca_1 - c_1a} = \frac{z}{ab_1 - a_1b}.$$

Eliminate z from the given equations.

$$ac_1x + bc_1y + cc_1z = 0,$$
$$a_1cx + b_1cy + cc_1z = 0.$$
$$\therefore \quad x(ac_1 - a_1c) + y(bc_1 - b_1c) = 0,$$
$$\therefore \quad \frac{x}{y} = \frac{bc_1 - b_1c}{ca_1 - c_1a} \text{ or } \frac{x}{bc_1 - b_1c} = \frac{y}{ca_1 - c_1a}.$$

Similarly, by eliminating x we get

$$\frac{y}{ca_1 - c_1a} = \frac{z}{ab_1 - a_1b}.$$

$$\therefore \quad \frac{x}{bc_1 - b_1c} = \frac{y}{ca_1 - c_1a} = \frac{z}{ab_1 - a_1b}, \text{ as required.}$$

In particular cases it is convenient to be able to write down the ratios without performing the intermediate eliminations. This is not difficult to do if we observe how the denominators of the fractions are obtained from the coefficients of the given equations.

Denominator of x	*Denominator of y*	*Denominator of z*
$b \times c$	$a \times c$	$a \times b$
$b_1 \times c_1$	$a_1 \times c_1$	$a_1 \times b_1$
$bc_1 - b_1c$	$ca_1 - c_1a$	$ab_1 - a_1b$

Note that in obtaining the denominator of y, the difference of the products is taken in the opposite order from the other two cases.

This may be referred to as the **rule of cross multiplication.**

Ex. **1.**—If $\left.\begin{aligned} x - 5y + 2z &= 0, \\ 7x + 9y - 6z &= 0, \end{aligned}\right\}$ find the ratios $x : y : z$.

By cross multiplication,

$$\frac{x}{(-5)(-6) - 2 \cdot 9} = \frac{y}{2 \cdot 7 - 1(-6)} = \frac{z}{1 \cdot 9 - (-5)7},$$

or
$$\frac{x}{12} = \frac{y}{20} = \frac{z}{44},$$

or
$$\frac{x}{3} = \frac{y}{5} = \frac{z}{11}.$$

Ex. **2.**—Solve
$$3x - 2y + 4z = 0, \quad . \quad . \quad . \quad . \quad (1)$$
$$2x - 3y + z = 0, \quad . \quad . \quad . \quad . \quad (2)$$
$$x^2 + y^2 + z^2 = 150. \quad . \quad . \quad . \quad . \quad (3)$$

From (1) and (2) by cross multiplication,

$$\frac{x}{10} = \frac{y}{5} = \frac{z}{-5} \quad \text{or} \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{-1}.$$

Le $\frac{x}{2} = \frac{y}{1} = \frac{z}{-1} = k, \quad \therefore \ x = 2k, y = k, z = -k.$

Substituting in (3), $4k^2 + k^2 + k^2 = 150, \quad \therefore \ k = \pm 5.$

$$\therefore \ x = \pm 10, \ y = \pm 5, \ z = \mp 5.$$

Ex. 3.—Solve

$$2xy + yz - 3zx = 0, \quad . \ . \ . \quad (1)$$
$$18xy - 5yz - 6zx = 0, \quad . \ . \ . \quad (2)$$
$$4x - 5y + 7z = 21. \quad . \ . \ . \quad (3)$$

From (1) and (2), $\frac{xy}{-21} = \frac{yz}{-42} = \frac{zx}{-28}$ or $\frac{xy}{3} = \frac{yz}{6} = \frac{zx}{4}.$

Since $\frac{xy}{3} = \frac{yz}{6} \ \therefore \ \frac{x}{1} = \frac{z}{2}$, and $\frac{xy}{3} = \frac{zx}{4} \ \therefore \ \frac{y}{3} = \frac{z}{4}$,

$$\therefore \ \frac{x}{2} = \frac{y}{3} = \frac{z}{4} = k.$$

Substituting in (3), $8k - 15k + 28k = 21, \quad \therefore \ k = 1,$

$$\therefore \ x = 2, \ y = 3, \ z = 4.$$

EXERCISE 14

1. If a, b, x are positive, show that $a + bx : b + ax$ lies between $a : b$ and $b : a$.

2. Show that any ratio is increased by adding to its terms the corresponding terms of a greater ratio.

3. If a is positive, which is the greater ratio

$$1 + 2a : 1 + 3a \text{ or } 1 + 3a : 1 + 4a ?$$

4. If a and b have like signs, which is the greater ratio

$$a^2 - ab + b^2 : a^2 + ab + b^2 \text{ or } a^4 - a^2b^2 + b^4 : a^4 + a^2b^2 + b^4 ?$$

5. If $3x + 4y : 7x - 3y = 18 : 5$, find $x : y$.

Solve for the ratio $x : y$.

6. $ax + by = cx + dy$.

7. $3x^2 - 10xy + 3y^2 = 0$.

8. $6x^2 + 4y^2 = 11xy$.

9. $5x^2 = 3xy + y^2$.

Find the ratios of x, y, z in each of the following :

10. $x + y + z = 0$, $2x + y + 4z = 0$.

11. $2x - y - z = 0$, $5x - y - 3z = 0$.

12. $x + y + z = 0$, $ax + by + cz = 0$.

13. $3x - 2y + 4z = 5x + 2y + 2z = 4x - 4y + 4z$.

14. $\frac{1}{2}(x + y) = \frac{1}{5}(2y + 7z) = \frac{1}{3}(2x + z)$.

15. If $x + y + z = a + b + c$ and $ax + by + cz = ab + bc + ca$, find the ratios of $x - b$, $y - c$, $z - a$.

16. Show that the equations $x + y - z = 0$, $7x - 9y + 3z = 0$, and $2x + 5y - 3z = 0$ cannot all be true at the same time.

17. If $\frac{3x + 2y}{3} = \frac{x + 2y}{2}$, then each ratio $= \frac{17x + 22y}{25}$.

18. If $\frac{2a + 3b}{9} = \frac{4b + 3c}{11} = \frac{2c + 2a}{5}$, then each $= \frac{10a + 11b - 7c}{30}$.

19. If 4 men and 3 women earn as much as 16 boys, and 6 men and 5 boys earn as much as 10 women, find the ratios of the earnings of a man, woman and boy.

20. If $\frac{x}{b - c} + \frac{y}{c - a} + \frac{z}{a - b} = \frac{x}{a(b - c)} + \frac{y}{b(c - a)} + \frac{z}{c(a - b)} = 0$, find the ratios of x, y, z.

Solve the following equations :

21. $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, $xy + yz - zx = 40$.

22. $3x + y - z = x - 5y + z = 0$, $x^2 + 2y^2 + 3z^2 = 51$.

23. $2x - y = 5z$, $3x + 15z = 2y$, $xyz = 9$.

24. $\frac{1}{2}yz = \frac{1}{3}zx = \frac{1}{6}xy$, $x + y + z = 36$.

25. $2x - 3y + 4z = 0$, $3x - y - 2z = 0$, $x^3 + y^3 + z^3 = 5439$.

26. $x + y + z = cx + ay + bz = 0$, $x^2 - y^2 + z^2 = 2(a^2 + bc - ab - ac)$.

27. $x^2 + y^2 - z^2 = 10x^2 - 15y^2 + 6z^2 = 2x^2 + 3y^2 + z^2 = 18x + 10y - 3z + 12$.

28. $ax + by + cz = x + y + z = a^2x + b^2y + c^2z + (a - b)(b - c)(c - a) = 0$.

29. $x + y + z = a + b + c$, $ax + by + cz = ab + bc + ca$, $x(b - c) + y(c - a) + z(a - b) = 0$.

42. Eliminations.

Ex. 1.—Eliminate x and y from $ax + by = 0$, . . . (1)
and $mx^2 + nxy + py^2 = 0$. . . (2)

(1) may be written $\frac{x}{y} = -\frac{b}{a}$.

(2) may be written $m\left(\frac{x}{y}\right)^2 + n\left(\frac{x}{y}\right) + p = 0$.

Substituting $\frac{x}{y} = -\frac{b}{a}$, $m\left(-\frac{b}{a}\right)^2 + n\left(-\frac{b}{a}\right) + p = 0$,

or $mb^2 - nab + pa^2 = 0$.

Therefore the result of eliminating x and y from the given equations is $mb^2 - nab + pa^2 = 0$. This is the condition that the equations may be satisfied by the same pair of values of x and y other than zero.

Ex. 2.—Eliminate x from $a\,x^2 + b\,x + c = 0$,
and $a_1x^2 + b_1x + c_1 = 0$.

By cross multiplication, $\frac{x^2}{bc_1 - b_1c} = \frac{x}{ca_1 - c_1a} = \frac{1}{ab_1 - a_1b}$.

$\therefore \quad \left(\frac{ca_1 - c_1a}{ab_1 - a_1b}\right)^2 = \frac{bc_1 - b_1c}{ab_1 - a_1b}$.

$\therefore \quad (ca_1 - c_1a)^2 = (bc_1 - b_1c)(ab_1 - a_1b)$.

Ex. 3.—What is the condition that the equations

$$a\,x + b\,y + c\,z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$a_1x + b_1y + c_1z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$a_2x + b_2y + c_2z = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

may be satisfied by the same values of x, y, z other than zero?

From (1) and (2), $\frac{x}{bc_1 - b_1c} = \frac{y}{ca_1 - c_1a} = \frac{z}{ab_1 - a_1b}$.

Dividing each term of (3) by one of these equal fractions,

$$a_2(bc_1 - b_1c) + b_2(ca_1 - c_1a) + c_2(ab_1 - a_1b) = 0.$$

This is the required condition. If it is not satisfied, the only roots of the equations are $x = y = z = 0$.

If this condition is satisfied, any values of x, y and z which satisfy any two of the equations will also satisfy the third.

This condition is sometimes called the **eliminant** of the given equations.

EXERCISE 15

Eliminate x and y from the equations :

1. $3x - 4y = 0$, $ax^2 + bxy + cy^2 = 0$.

2. $a(x + y) = b(x - y)$, $x^2 + y^2 = 2cxy$.

3. $3(x + y) = 4(x - y) = c$, $x^2 + y^2 = 2d^2$.

Eliminate x, y, z from

4. $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$.

5. $a(x + y) = z$, $b(y + z) = x$, $c(z + x) = y$.

6. $ax + by + cz = 0$, $lx + my + nz = 0$, $x^2 + y^2 = 2z^2$.

7. Eliminate x from $x^2 + bx + c = 0$ and $x^2 + b_1x + c_1 = 0$.

8. Eliminate x from $ax^3 + bx + c = 0$ and $a_1x^3 + b_1x + c_1 = 0$.

9. If $x^2 + 2x - 1 = 0$ and $x^2 + ax - 2 = 0$ have a common root, show that $a = 3 \pm \sqrt{2}$.

10. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$, show that

$$\frac{x^2}{1 - a^2} = \frac{y^2}{1 - b^2} = \frac{z^2}{1 - c^2}.$$

(Write down the ratios of x, y, z from the equations in pairs. Then multiply equal ratios.)

11. If $ax + hy + gz = 0$, $hx + by + fz = 0$, $gx + fy + cz = 0$, prove that

(1) $$\frac{x^2}{bc - f^2} = \frac{y^2}{ca - g^2} = \frac{z^2}{ab - h^2}.$$

(2) $(bc - f^2)(ca - g^2)(ab - h^2) = (fg - ch)(gh - af)(hf - bg)$.

12. If $cx + ay + bz = 0$, $bx + cy + az = 0$, show that

(1) $\frac{1}{a}(x^2 - yz) = \frac{1}{b}(y^2 - zx) = \frac{1}{c}(z^2 - xy)$,

(2) $(b - c)(x^2 - yz) + (c - a)(y^2 - zx) + (a - b)(z^2 - xy) = 0$.

43. If any Number of Ratios are Equal, any one of them is Equal to the Ratio of the Sum of all the Antecedents to the Sum of all the Consequents.

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$, and let each ratio $= k$,

then $a = bk$, $c = dk$, $e = fk \ldots$

$$\therefore \frac{a + c + e \ldots}{b + d + f \ldots} = \frac{bk + dk + fk \ldots}{b + d + f \ldots} = k,$$

which proves the proposition.

This theorem is a particular case of the following general theorem :

If $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots,$$

then each ratio is equal to

$$\left(\frac{pa^n + qc^n + re^n \ldots}{pb^n + qd^n + rf^n \ldots}\right)^{\frac{1}{n}}.$$

Prove this theorem in the same manner as the particular case.

Thus, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

each ratio is equal to

$$\frac{a + c - e}{b + d - f}, \quad \frac{a - c + e}{b - d + f}, \quad \frac{2a - 3c + 5e}{2b - 3d + 5f},$$

and so on.

Ex. 1.—If $$\frac{b + c}{x} = \frac{c + a}{y} = \frac{a + b}{z},$$

then $$\frac{a}{y + z - x} = \frac{b}{z + x - y} = \frac{c}{x + y - z}.$$

To obtain the denominator $y + z - x$ from the denominators of the given fractions we must add the second and third and subtract the first. The other denominators of the required fractions may be obtained in a similar way.

Let each of the given fractions $= k$,

then $$k = \frac{(c + a) + (a + b) - (b + c)}{y + z - x} = \frac{2a}{y + z - x}.$$

Similarly, $$k = \frac{2b}{z + x - y} \text{ and } k = \frac{2c}{x + y - z},$$

$$\therefore \frac{a}{y + z - x} = \frac{b}{z + x - y} = \frac{c}{x + y - z}$$

as required.

Ex. 2.—If $$\frac{2y + 2z - x}{a} = \frac{2z + 2x - y}{b} = \frac{2x + 2y - z}{c},$$

show that $$\frac{z}{2a + 2b - c} = \frac{x}{2b + 2c - a} = \frac{y}{2c + 2a - b}.$$

How can we obtain the first denominator, $2a + 2b - c$, from the denominators of the given fractions?

$$k = \frac{2(2y + 2z - x) + 2(2z + 2x - y) - (2x + 2y - z)}{2a \quad + \quad 2b \quad - \quad c} = \frac{9z}{2a + 2b - c}.$$

Similarly, $$k = \frac{9x}{2b + 2c - a} = \frac{9y}{2c + 2a - b},$$

$$\therefore \frac{z}{2a + 2b - c} = \frac{x}{2b + 2c - a} = \frac{y}{2c + 2a - b}$$

as required.

Ex. 3.—If $$\frac{x + y}{pa + qb} = \frac{y + z}{pb + qc} = \frac{z + x}{pc + qa},$$

show that $$\frac{2(x + y + z)}{a + b + c} = \frac{x(b + c) + y(c + a) + z(a + b)}{ab + bc + ca}.$$

$2(x + y + z)$ is evidently the sum of the given numerators,

$$\therefore k = \frac{(x + y) + (y + z) + (z + x)}{(pa + qb) + (pb + qc) + (pc + qa)} = \frac{2(x + y + z)}{(p + q)(a + b + c)}.$$

Since $x(b+c)+y(c+a)+z(a+b)=c(x+y)+a(y+z)+b(x+y)$, it is evident that the numerators and denominators of the given fractions should be multiplied by c, a, b, respectively, before adding.

$$k=\frac{c(x+y)+a(y+z)+b(z+x)}{c(pa+qb)+a(pb+qc)+b(pc+qa)}=\frac{c(x+y)+a(y+z)+b(z+x)}{(p+q)(ab+bc+ca)}.$$

$$\therefore\quad \frac{2(x+y+z)}{(p+q)(a+b+c)}=\frac{c(x+y)+a(y+z)+b(z+x)}{(p+q)(ab+bc+ca)},$$

$$\therefore\quad \frac{2(x+y+z)}{a+b+c}=\frac{x(b+c)+y(c+a)+z(a+b)}{ab+bc+ca}.$$

Ex. 4.—If $\dfrac{a}{x+y-z}=\dfrac{b}{y+z-x}=\dfrac{c}{z+x-y}$,

show that $a(x-y)+b(y-z)+c(z-x)=0$.

Here $k=\dfrac{a(x-y)+b(y-z)+c(z-x)}{(x+y-z)(x-y)+(y+z-x)(y-z)+(z+x-y)(z-x)}$

$$=\frac{a(x-y)+b(y-z)+c(z-x)}{0}.$$

The value of this fraction is indefinitely great or ∞, unless the numerator is 0. It is assumed that the given fractions are finite in value, that is, they are not either 0 or ∞.

$$\therefore\quad a(x-y)+b(y-z)+c(z-x)=0.$$

44. If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots$ are unequal fractions, then the fraction

$$\frac{a+c+e\ \ldots}{b+d+f\ \ldots}$$

lies between the greatest and least of the given fractions.

Let $\frac{a}{b}$ be the greatest of the given fractions and let $\frac{a}{b}=k$,

then $$\frac{c}{d}<k, \frac{e}{f}<k \ldots$$

$$\therefore \quad a = bk,\ c < dk,\ e < fk \ldots$$

$$a + c + e + \ldots < bk + dk + fk + \ldots$$

$$\therefore \quad \frac{a + c + e + \ldots}{b + d + f + \ldots} < k,$$

that is, less than the greatest fraction.

In the same way we may show that it is greater than the least of the given fractions.

Deduce the theorem in Art. 43 from this theorem.

45. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. Prove this theorem.

Ex.—Solve $\frac{(2x+3)^3 + (x-1)^3}{(2x+3)^3 - (x-1)^3} = \frac{65}{63}$.

Adding and subtracting, $\frac{2(2x+3)^3}{2(x-1)^3} = \frac{128}{2}$, $\therefore \frac{(2x+3)^3}{(x-1)^3} = 64$,

$$\therefore \quad \frac{2x+3}{x-1} = 4, \qquad \therefore \quad x = 3\tfrac{1}{2}.$$

EXERCISE 16

1. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that each fraction

$$= \frac{2a + 3c - 4e}{2b + 3d - 4f} = \sqrt[3]{\frac{a^3 - c^3 + e^3}{b^3 - d^3 + f^3}}.$$

2. If $\frac{x}{a+b-c} = \frac{y}{b+c-a} = \frac{z}{c+a-b}$, show that

$$x(a-b) + y(b-c) + z(c-a) = 0.$$

3. If $\frac{a}{x+y-z} = \frac{b}{y+z-x} = \frac{c}{z+x-y}$, show that $\frac{x}{a+c} = \frac{y}{b+a} = \frac{z}{c+b}$.

4. If $\frac{a+b}{a-b} = \frac{b+c}{b-c} = \frac{c+d}{c-d}$, then each $= \frac{a+d}{a-2b+2c-d}$.

5. If $\frac{x}{p} = \frac{y}{q} = \frac{z}{r}$ where p, q, r are not zero, and if $p + q + r = 0$, show that $x + y + z = 0$.

6. If $\dfrac{a+b+c+d}{a+b-c-d} = \dfrac{a-b+c-d}{a-b-c+d}$, show that $\dfrac{a}{b} = \dfrac{c}{d}$.

7. Solve (1) $\dfrac{ax+b+c}{ax-b+c} = \dfrac{bx+c+a}{bx+c-a}$, (2) $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}} = c$.

8. Solve $\dfrac{2x^3+x^2-3x+4}{2x^3-x^2-3x-4} = \dfrac{x^4+4x^3-6x-16}{16+4x^3-6x-x^4}$.

9. If $x+y+z = ax+by+cz = x^2+y^2-z^2 = 0$, then $c = a$ or b.

10. If $a-x : b-x$ is the duplicate of $a : b$, show that $2x$ is the harmonic mean between a and b.

11. Solve $\dfrac{2x+3y-4z}{x+5} = \dfrac{3x+4y-2z}{5x} = \dfrac{4x+2y-3z}{4x-1} = \dfrac{x+y-z}{6}$.

12. If $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\dfrac{e}{f}$ be unequal, then $\dfrac{la+mc+ne}{lb+md+nf}$ is intermediate between the greatest and least of these fractions.

13. If four quantities are proportionals, show that the sum of the greatest and least is greater than the sum of the other two.

14. If $a : b = c : d$, of which a is the greatest, show that

$$a^2 + d^2 > b^2 + c^2.$$

15. If four positive quantities are in continued proportion, show that the difference between the first and last is at least three times as great as the difference between the other two.

16. If $a : b = c : d$ and no two are equal, show that there is no finite quantity, which being added to each, will leave the sums proportional.

17. If m gold coins are equal in weight to n silver coins, and p of the former are equal in value to q of the latter, compare the values of equal weights of gold and silver.

18. If $(a^2+b^2+c^2)(x^2+y^2+z^2) = (ax+by+cz)^2$,

show that $(ay-bx)^2 + (bz-cy)^2 + (cx-az)^2 = 0$

and deduce that $x : a = y : b = z : c$.

19. If $\dfrac{ad-bc}{a-b-c+d} = \dfrac{ac-bd}{a-b+c-d}$ and a, b, c, d are unequal,

then each fraction $= \dfrac{a+b}{2} = \dfrac{c+d}{2} = \dfrac{a+b+c+d}{4}$.

20. If $\frac{a+b}{a-b} = \frac{2(b+c)}{3(b-c)} = \frac{4(c+a)}{5(c-a)}$, then $27a + 25b + 22c = 0$.

21. If $\frac{a^2-bc}{a-abc} = \frac{b^2-ca}{b-abc}$ and a and b are unequal, show that

$$a+b+c = \frac{ab+bc+ca}{abc} = \frac{c^2-ab}{c-abc}.$$

22. If $\frac{a}{x^2-yz} = \frac{b}{y^2-zx} = \frac{c}{z^2-xy}$, then

$$ax + by + cz = (a+b+c)(x+y+z).$$

23. If $\frac{a(y+z)}{x} = \frac{b(z+x)}{y} = \frac{c(x+y)}{z}$, then $\frac{a-b}{x^2-y^2} = \frac{b-c}{y^2-z^2} = \frac{c-a}{z^2-x^2}$.

24. If $x + y : 3a - b = y + z : 3b - c = z + x : 3c - a$, prove that

$$(x+y+z)(a^2+b^2+c^2) = (a+b+c)(ax+by+cz).$$

25. If $\frac{ax+by}{a^2+b^2} = \frac{by+cz}{b^2+c^2} = \frac{cz+ax}{c^2+a^2}$, then $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

26. If the three fractions $\frac{2x-3y}{3z+y}$, $\frac{z-y}{z-x}$, $\frac{x+3z}{2y-3x}$ are all equal, each is equal to $\frac{x}{y}$ and hence $x = y$ or $z = x + y$.

27. If $a + b$, $b + c$, $c + d$, $d + a$ are proportionals, prove that

$$a = c \text{ or } a + b + c + d = 0.$$

28. If $\frac{a}{b+c-a} = \frac{b}{c+a-b} = \frac{c}{a+b-c}$, prove that each fraction equals 1 or $-\frac{1}{2}$.

29. If $b^2 + bx + x^2 : a^2 + ay + y^2 = b^2 - bx + x^2 : a^2 - ay + y^2$, prove that $x : b = a : y$, or $x : b = y : a$.

CHAPTER VI

VARIATION

46. If a yard of a certain quality of carpet costs \$2, then 2 yards of the same quality will cost \$4, 3 yards will cost \$6, and so on. That is, when the number of yards is multiplied by any number the total cost is multiplied by the same number.

Although there is a change in the number of yards bought and in the total cost, there is no change in the ratio of the measures of the total cost and the number of yards.

When two quantities are so related, we say that either quantity varies as the other. Here the cost varies as the number of yards and the number of yards varies as the cost.

47. Direct Variation. If two variable quantities a and b are so related that the ratio of simultaneous values of a and b is constant, then either quantity is said to **vary directly** as the other.

The symbol $\propto$ is used for the words "varies as."

Thus, $a \propto b$ is read "a varies as b."

From the definition it follows that if $a \propto b$, then $\frac{a}{b} = m$, or $a = mb$, where m is a constant which is independent of the values of a and b.

This conclusion is of great importance as it allows us to change from a statement of variation to an ordinary equation.

Conversely, if $x = my$, where m is a constant, then $\frac{x}{y} = m$ and the ratio of $x : y$ is a constant.

$$\therefore \quad x \propto y, \text{ by definition.}$$

48. Inverse Variation. When a varies as the reciprocal of b, a is said to **vary inversely** as b.

Hence if a varies inversely as b,

$$a = m \cdot \frac{1}{b} \text{ where } m \text{ is a constant.}$$

When a is said to vary as b, it is understood that a varies directly as b.

The symbol m is sometimes called the **variation constant.**

The following are simple illustrations of variation :

The amount of work done varies as the number of men employed.

The time required to complete a certain work varies inversely as the number of men employed.

The volume of a gas varies as the absolute temperature.

The volume of a gas varies inversely as the pressure.

The circumference of a circle varies as the radius.

The volume of a sphere varies as the cube of the radius.

Ex. 1.—If $x \propto y$ and when y is 10 x is 3, find the definite relation connecting x and y.

Since $x \propto y$, let $x = my$ where m is constant.

Now when $y = 10$, $x = 3$, therefore $m = \frac{3}{10}$.

$\therefore \quad x = \frac{3}{10}y$ is the required relation.

If now the value of x or y is given, the corresponding value of the other can be found.

Ex. 2.—The area of a circle varies as the square on its radius. When the radius is $3\frac{1}{2}$ in. the area is $38\frac{1}{2}$ sq. in. Find the area when the radius is 14 in.

Let r be the measure of the radius.

Let a be the measure of the area.

Then $a \propto r^2$, $\therefore \quad a = mr^2$, where m is constant.

If we substitute $a = 38\frac{1}{2}$, $r = 3\frac{1}{2}$ in this equation we find that $m = 3\frac{1}{7}$.

$$\therefore \quad a = 3\tfrac{1}{7}r^2.$$

Now if $r = 14$, $a = 3\frac{1}{7} \times 14^2 = 616$.

$\therefore$ the area = 616 sq. in.

Ex. 3.—The expenses of an entertainment are partly constant and partly vary as the number of guests. For 80 guests the cost is \$160 and for 120 guests the cost is \$220; find the cost for 200 guests.

Let \$$x$ be the total cost, \$$c$ the constant part of the expense, and n the number of guests.

Then $x = c + mn$, where c and m are constants.

When $n = 80$, $x = 160$, $\therefore \quad 160 = c + 80m$.

When $n = 120$, $x = 220$, $\therefore \quad 220 = c + 120m$.

Solving we find $m = 1\frac{1}{2}$, $c = 40$.

$$\therefore \quad x = 40 + 1\tfrac{1}{2}n.$$

Now if $n = 200$, $x = 340$.

$\therefore$ the cost for 200 guests is \$340.

Ex. 4.—If $x \propto y$, show that $x^2 - xy \propto y^2 - 3x^2$.

Since $x \propto y$, let $x = my$ where m is constant.

Then $\dfrac{x^2 - xy}{y^2 - 3x^2} = \dfrac{m^2y^2 - my^2}{y^2 - 3m^2y^2} = \dfrac{m^2 - m}{1 - 3m^2}$, which is constant since its value does not depend upon the values of x and y.

$$\therefore \quad x^2 - xy \propto y^2 - 3x^2.$$

In the same way it may be shown that if $x \propto y$, then any homogeneous function of x and y varies as any other homogeneous function of the same degree.

EXERCISE 17

1. Express the following statements as equations :

(1) a varies as b. (2) y varies inversely as x. (3) $A \propto R^2$. (4) s varies as the square of t. (5) The cube of y varies inversely as the square root of x.

2. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. How does V vary with r? How does r vary with V?

3. If x varies as the square of y and if $x = 100$ when $y = 2$, find the variation constant and find the value of x when $y = 3$.

4. If x varies inversely as the square of y and $x = 3a$ when $y = 4a$, find the definite relation between x and y.

5. If x varies inversely as $\dfrac{y - z}{yz}$ and if $x = 5$ when $y = 7$ and $z = 2$, find the equation connecting x, y, z.

6. If the following are simultaneous values of x and y, how does y vary with respect to x?

$$x = 1,\ 1\tfrac{1}{2},\ 2,\ 2\tfrac{1}{2},\ 3.$$
$$y = 4,\ 9,\ 16,\ 25,\ 36.$$

What would x be when y is 6·25?

7. If $a \propto b$, show that $a^3 + b^3 \propto 3a^2b + 2ab^2$.

8. If $x + y \propto x - y$, show that $x \propto y$ and $x^2 + y^2 \propto 3xy$.

9. The weight of a cube varies as the cube of its edge. If the weight is 32 lb. when the edge is 4 in., find the weight when the edge is 5 in.

10. If $x^2 \propto y$ and $y \propto z^2$, show that $x \propto z$.

11. If the square of y varies as the cube of z and z varies inversely as x, show that xy varies inversely as the square root of x.

12. If v varies as w^2, w^3 as x^4, x^5 as y^6 and y^7 as z^4, show that the product $\dfrac{v}{z} \cdot \dfrac{w}{z} \cdot \dfrac{x}{z} \cdot \dfrac{y}{z}$ does not vary at all.

13. If y is equal to the sum of two quantities, one of which varies as x^2 and the other inversely as x, and when $x = 1$, $y = 12$ and also when $x = 2$, $y = 20$, find the value of y when $x = \frac{1}{2}$.

14. The sum of the squares of two quantities varies as their sum, and when the quantities are equal, but not zero, each of them is 2. Find the value of one when the other is $\frac{1}{5}$.

15. The area of a circle varies as the square of the radius. If the area is $9\frac{6}{8}$ sq. ft. when the radius is 1 ft. 9 in., find the area when the radius is 5 ft. 3 in., and find the radius when the area is $38\frac{1}{2}$ sq. ft.

16. The distance that a body falls from rest varies as the square of the time of falling. If a body falls 256 ft. in 4 sec., find (*a*) how far it would fall in 10 sec., (*b*) how far it would fall in the first second, in the tenth second, (*c*) in what time it would fall 3200 ft.

17. The attraction between two objects varies inversely as the square of the distance between them. If the attraction is 10 when the objects are 5 in. apart, what is the attraction when they are 15 in. apart?

18. The volume of a sphere varies as the cube of its radius. Three metal spheres of radii 3, 4, 5 inches are melted into a single sphere. Find its radius.

19. The time of the swing of a pendulum varies as the square root of its length. If a pendulum 1 metre in length swings once in a second, find the length of a pendulum which swings 40 times in a minute.

20. The velocity of a particle falling vertically varies as the square root of the distance through which it has fallen from rest. If the velocity of a particle which has fallen 25 ft. is 40 ft. per sec., what was the velocity when it had fallen 2·56 ft.?

21. The square of the time of a planet's revolution about the sun varies as the cube of its mean distance from the sun. Jupiter is 5·2 times as far from the sun as the earth is. Find, in years, Jupiter's time of revolution.

22. If $H \propto D^{\frac{2}{3}}v^3$, and if D is 1810 and v is 10 when H is 620, use logarithms to find H if D is 2100 and v is 13.

23. The expenses of a charitable institution are partly constant and partly vary as the number of inmates. When the number of inmates is 970 and 3000 the expenses are respectively $\$561\frac{2}{3}$ and \$900. What are the expenses when the number of inmates is 1500?

24. The value of a diamond varies as the square of its weight. If a diamond worth \$600 is broken into two parts whose weights are as 2 : 3, by how much has its value been reduced?

25. It is known that the cost per hour of running a steamboat consists of two parts, one of which is constant and the other varies as the cube of the boat's speed. The cost of making a 60-mile trip at 10 miles an hour is \$150, and at 12 miles an hour is \$143·20. Find the cost at 15 miles an hour.

49. More than Two Variables. Geometrical illustration. It is proven in geometry that

(a) rectangles with equal heights are in the ratio of their bases,

(b) rectangles with equal bases are in the ratio of their heights.

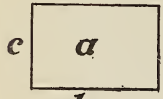


Fig. 1.

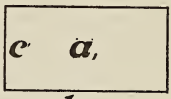


Fig. 2.

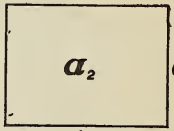


Fig. 3.

In fig. 1, the measure of the area is a, of the base b and of the height c. In fig. 2, c is unchanged while b has changed to b_1, which has caused the area to change from a to a_1.

Since by theorem (a) these changes are in the same ratio

$$\frac{a}{a_1} = \frac{b}{b_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In fig. 3, the base is unchanged while c has changed to c_1, which has caused the area to change from a_1 to a_2.

$$\therefore \text{ by theorem } (b), \frac{a_1}{a_2} = \frac{c}{c_1}. \quad . \quad . \quad . \quad (2)$$

Multiplying (1) by (2), $\frac{a}{a_2} = \frac{bc}{b_1 c_1}$.

$\therefore$ the change in a is proportional to the change in bc.

$$\therefore \quad a \propto bc.$$

Therefore the area of a rectangle varies as the product of the base and height when the base and height vary.

50. If a varies as b when c is constant and a varies as c when b is constant, then a will vary as bc when b and c vary.

Let (a_1, b_1, c), (a_2, b_1, c_1) be two simultaneous sets of values of a, b, c. In the first set c remains constant and in the second b is constant.

Then since a varies as b

$$\frac{a}{a_1} = \frac{b}{b_1}. \quad \ldots\ldots\ldots \quad (1)$$

And since a varies as c

$$\frac{a_1}{a_2} = \frac{c}{c_1}. \quad \ldots\ldots\ldots \quad (2)$$

Multiplying (1) and (2) $\frac{a}{a_2} = \frac{bc}{b_1c_1}$.

$$\therefore \quad a \propto bc.$$

It will be seen that the reasoning here is the same as in the geometrical illustration preceding.

51. Joint Variation. When a varies as the product of two or more quantities it is said to **vary jointly** as those quantities.

Thus, the area of a rectangle varies jointly as the base and height.

The amount of work done varies jointly as the number of men working and the number of days they work.

The interest varies jointly as the principal, time and rate.

If z varies directly as x and inversely as y, then z varies as $x \times \frac{1}{y}$ or $z \propto \frac{x}{y}$.

Ex. 1.—The volume of a cone varies as the height when the area of the base is constant and as the square on the radius of the base when the height is constant. If the height

is 12 in. and the radius of the base $3\frac{1}{2}$ in., the volume is found to be 154 cu. in. Find the formula for the volume of a cone.

Let V (cu. in.) be the volume, h (in.) the height and r (in.) the radius of the base.

Then since V varies jointly as h and r^2,

$$V = mhr^2, \text{ where } m \text{ is a constant.}$$

Substituting $V = 154$, $h = 12$, $r = 3\frac{1}{2}$ we find $m = \frac{22}{7}$.

$$\therefore \quad V = \tfrac{22}{7}hr^2 \text{ is the required formula.}$$

Ex. 2.—If t sec. is the time taken by a body to slide down a smooth inclined plane whose length is l ft. and vertical height h ft., it can be shown that t varies directly as l and inversely as $\sqrt{h}$. If $t = 1$ when $l = 8$ and $h = 4$, find h when $l = 3$ and $t = \frac{1}{2}$.

Here t varies jointly as l and $\dfrac{1}{\sqrt{h}}$,

$$\therefore \quad t = m \cdot \frac{l}{\sqrt{h}}, \text{ where } m \text{ is a constant.}$$

Substituting $t = 1, l = 8, h = 4$ we find $m = \frac{1}{4}$,

$$\therefore \quad t = \tfrac{1}{4}\frac{l}{\sqrt{h}}.$$

Now when $l = 3$ and $t = \frac{1}{2}$, $h = 2\frac{1}{4}$.

EXERCISE 18

1. If a varies as b and c jointly, while b varies as d^2, and c varies inversely as a, show that a varies as d.

2. If $x \propto y$ when z is constant and $x \propto z$ when y is constant, then y varies inversely as z when x is constant.

3. If a varies directly as b and inversely as c and when $a = 2$, $b = 3$, $c = 4$, find b when $a = 6$, $c = 3$.

4. If y is equal to the sum of three quantities of which the first is constant, the second varies as x and the third as x^2 and when $x = 1, 2, 3$, $y = 6, 11, 18$, express y in terms of x.

5. If $a^2 - b^2 \propto c^2$, and if $c = 2$ when $a = 5$ and $b = 3$, show that b is a mean proportional between $a - 2c$ and $a + 2c$.

6. If $x^2 + y^2 \propto xy$, show that $\frac{x}{y}$ is constant and that therefore $x \propto y$.

7. If $x^3 + y^3 \propto x^2y + xy^2$, show that $x^2 - y^2 \propto 2xy$.

8. If x varies as the sum of two quantities and x^2 varies as the sum of their squares, then x^2 will also vary as their product.

9. The volume of a gas varies as the absolute temperature and inversely as the pressure. If the volume is 78 cu. ft. when the absolute temperature is 273° and the pressure 14 lb. to the square inch, find the volume when the temperature is 300° and the pressure 20 lb.

10. The amount of illumination varies directly as the intensity of the light and inversely as the square of the distance from the light. What must be the intensity of a light to give at a distance of 75 feet three times the illumination of a light whose intensity is 10 and whose distance is 50 feet?

11. The resistance offered by a wire to an electric current varies as its length and inversely as the area of its cross section. If a copper wire $\frac{1}{8}$ of an inch in diameter has a resistance of 28 ohms per mile, find the resistance of a wire of the same material $\frac{1}{4}$ of an inch in diameter and 3 miles long.

12. The weight of a sphere varies as its density and as the cube of its radius. The densities of two spheres are as 2 : 3 and their radii as 16 : 7. If the weight of the first is 50 lb., find the weight of the second.

13. The volume of a cylinder varies as the height and the area of the base jointly. If the volume of the first of two cylinders is to that of the second as 11 : 8 and the height of the first is to that of the second as 3 : 4, and if the area of the base of the first is 16·5 sq. ft., find the area of the base of the second.

14. The attraction between two bodies varies directly as the product of their masses and inversely as the square of the distance between their centres. What would 1 lb. weigh on the surface of a planet of the same density as the earth, but of 1·5 times the diameter?

15. The velocity of a railway train varies directly as the square root of the quantity of coal used per mile and inversely as the number of cars in the train. In a journey of 25 miles in 30 min. with 18 cars, 1000 lb. of coal is needed. How much coal will be consumed in a journey of 21 miles in 28 min. with 40 cars in the train?

16. If the quantities x, y, z be variable, but their sum constant, and if $(x - y + z)(x + y - z) \propto yz$, show that $x - y - z \propto yz$

17. The value of a coin varies jointly as the thickness and the square on the radius. Two coins have their thicknesses in the ratio of 25 : 32, find the ratio of their radii if the value of the first is twice that of the second.

18. Geometry teaches that the volume of a pyramid varies as its base when the height is constant and as the height when the base is constant. A cube may be divided into six equal pyramids, each having a side of the cube as base, and its vertex at the centre of the cube. Hence obtain the general formula for the volume of a pyramid.

19. The value of a diamond varies as the square of its weight. A diamond worth \$980 was broken into two parts, the total value being thus reduced to \$500. Into what fractional parts was it broken?

20. An engine without a train can go 24 miles per hour, but with cars attached its speed is diminished by a quantity which varies as the square root of the number of cars. With 4 cars the speed is 20 miles per hour. Find the greatest number of cars that the engine can move.

21. For printing a circular the estimate is 90c. for 100 copies and \$1·20 for 200 copies. Assuming that each estimate consists of (1) a charge for setting up the type which is independent of the number of copies printed, and (2) a charge for printing and paper which varies as the number of copies printed, find what the estimate should be for 1000 copies.

22. The light received from a distant source varies directly as the candle power of the source and inversely as the square of the distance. The light received from a certain star measures 8×10^{-5} units, and the light from the sun measures 10^6 units. The distance of the star is $4{\cdot}7 \times 10^{13}$ miles, and the distance of the sun $9{\cdot}4 \times 10^7$ miles. Compare the candle power of the star with that of the sun.

CHAPTER VII

SURDS

52. Rationalizing Factor. When a fraction has a surd denominator, the fraction may be simplified by multiplying the numerator and denominator by such a quantity as will make the denominator rational.

Thus,
$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

$$\frac{5\sqrt{2}}{2\sqrt{7}} = \frac{5\sqrt{2} \times \sqrt{7}}{2\sqrt{7} \times \sqrt{7}} = \frac{5\sqrt{14}}{14}.$$

$$\frac{c}{\sqrt{a} + \sqrt{b}} = \frac{c(\sqrt{a} - \sqrt{b})}{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})} = \frac{c(\sqrt{a} - \sqrt{b})}{a - b}.$$

The factor by which a surd quantity is multiplied so as to give a rational product is called the **rationalizing factor** of that quantity.

Thus, the rationalizing factor of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$ and of $\sqrt{a} - \sqrt{b}$ is $\sqrt{a} + \sqrt{b}$.

53. Conjugate Surds. Such surd expressions as $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ which differ only in the sign which connects their terms are called **conjugate surds.** The product of two conjugate surds is always a rational quantity.

Thus, the conjugate of $3 - \sqrt{5}$ is $3 + \sqrt{5}$ and their product is 4.

Ex.—Simplify: $\dfrac{4\sqrt{3}+4}{\sqrt{3}-1}-\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

$$\frac{4\sqrt{3}+4}{\sqrt{3}-1}=\frac{(4\sqrt{3}+4)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}=\frac{16+8\sqrt{3}}{2}=8+4\sqrt{3}.$$

$$\frac{2+\sqrt{3}}{2-\sqrt{3}}=\frac{(2+\sqrt{3})(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}=\frac{7+4\sqrt{3}}{1}=7+4\sqrt{3}.$$

$\therefore$ the given quantity $= (8+4\sqrt{3})-(7+4\sqrt{3})=1.$

54. Rationalizing Factor of $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})=(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$$
$$=a+b-c+2\sqrt{ab}.$$
$$(a+b-c+2\sqrt{ab})(a+b-c-2\sqrt{ab})$$
$$=(a+b-c)^2-(2\sqrt{ab})^2$$
$$=a^2+b^2+c^2-2ab-2ac-2bc.$$

This last product is seen to be rational, and therefore the rationalizing factor of $\sqrt{a}+\sqrt{b}+\sqrt{c}$

$$=(\sqrt{a}+\sqrt{b}-\sqrt{c})(a+b-c-2\sqrt{ab})$$
$$=(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}-\sqrt{c}).$$

Ex.—Rationalize the denominator of $\dfrac{1}{\sqrt{5}+\sqrt{3}+1}$.

$$\frac{1}{\sqrt{5}+\sqrt{3}+1}=\frac{\sqrt{5}+\sqrt{3}-1}{(\sqrt{5}+\sqrt{3}+1)(\sqrt{5}+\sqrt{3}-1)}=\frac{\sqrt{5}+\sqrt{3}-1}{7+2\sqrt{15}}$$
$$=\frac{(\sqrt{5}+\sqrt{3}-1)(7-2\sqrt{15})}{(7+2\sqrt{15})(7-2\sqrt{15})}=\frac{\sqrt{5}-3\sqrt{3}+2\sqrt{15}-7}{-11}.$$

If an approximate value of a quantity like $\dfrac{1}{\sqrt{5}+\sqrt{3}+1}$ is required, it may be found more readily from the given form than in the form with the rationalized denominator. Little value therefore attaches to the rationalizing of a denominator which contains more than two terms.

EXERCISE 19

Rationalize the denominator of :

1. $\dfrac{19}{\sqrt{50}+\sqrt{8}-\sqrt{3}}$.

2. $\dfrac{(2-\sqrt{3})(3-\sqrt{3})}{(2+\sqrt{3})(3+\sqrt{3})}$.

3. $\dfrac{2+2\sqrt{3}}{1-\sqrt{2}+\sqrt{3}}$.

4. $\dfrac{12}{3+\sqrt{5}-2\sqrt{2}}$.

5. $\dfrac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{10}-\sqrt{5}+\sqrt{3}}$.

6. $\dfrac{1}{\sqrt{a}+\sqrt{b}-\sqrt{a+b}}$.

7. Find the value of $\dfrac{1}{2+\sqrt{3}}$, if $\sqrt{3} = 1{\cdot}7320508$.

8. Simplify $(3+\sqrt{6}) \div (5\sqrt{3} - 2\sqrt{12} - \sqrt{32} + \sqrt{50})$.

9. If $\sqrt{2} = 1{\cdot}4142$, $\sqrt{3} = 1{\cdot}7321$, $\sqrt{6} = 2{\cdot}4495$, find to three decimal places the value of $\dfrac{1}{2\sqrt{3}+3\sqrt{2}-2}$, (*a*) by substituting in the given form, (*b*) by first rationalizing the denominator.

10. Find the value of $x^2 + xy + y^2$ when $x = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$, $y = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.

11. When $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$, find the value of $x^2 + x + \dfrac{1}{x} + \dfrac{1}{x^2}$.

12. Find the value of $\left(\dfrac{x+a}{x-a}\right)^2 - \dfrac{x}{3a}$ when $x = a(2\sqrt{3}+1)$.

13. If $x = \dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, $y = \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of $3x^2 - 5xy + 3y^2$.

14. If $2x = \sqrt{a} + \dfrac{1}{\sqrt{a}}$, find the value of $\dfrac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$.

55. A Quadratic Surd cannot be expressed as the Sum (or Difference) of a Rational Quantity and a Quadratic Surd.

Thus, $\sqrt{a}$ cannot be equal to $\sqrt{b} + c$ where $\sqrt{a}$ and $\sqrt{b}$ are surds and c is rational but not zero.

For, if possible, let $\sqrt{a} = \sqrt{b} + c$.

Then by squaring, $a = b + c^2 + 2c\sqrt{b}$,

$$\therefore \quad \sqrt{b} = \frac{a - b - c^2}{2c}.$$

But this is impossible, since $\sqrt{b}$ is a surd.

$\therefore$ $\sqrt{a}$ cannot be equal to $\sqrt{b} + c$, unless $c = 0$.

56. If $a + \sqrt{b} = c + \sqrt{d}$, where a and c are Rational and $\sqrt{b}$ and $\sqrt{d}$ are Surds, then $a = c$ and $b = d$.

If $a + \sqrt{b} = c + \sqrt{d}$, then $\sqrt{b} = \sqrt{d} + c - a$.

But this is impossible unless $c - a = 0$ or $c = a$, and when $c = a$ it follows that $b = d$.

$\therefore$ if $a + \sqrt{b} = c + \sqrt{d}$, $a = c$ and $b = d$.

Ex. 1.—Find the square root of $11 + 4\sqrt{7}$.

Let $\sqrt{11 + 4\sqrt{7}} = \sqrt{x} + \sqrt{y}$.

Squaring, $11 + 4\sqrt{7} = x + y + 2\sqrt{xy}$.

$\therefore$ $x + y = 11$ and $2\sqrt{xy} = 4\sqrt{7}$.

$\therefore$ $x + y = 11$ and $xy = 28$.

Solving for x and y, we get $x = 7$ or 4, $y = 4$ or 7.

$\therefore$ $\sqrt{11 + 4\sqrt{7}} = \sqrt{4} + \sqrt{7} = 2 + \sqrt{7}$.

Ex. 2.—Find the square root of $a + \sqrt{b}$.

Let $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$.

Squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$,

$\therefore$ $x + y = a$ and $4xy = b$,

$\therefore$ $(x - y)^2 = (x + y)^2 - 4xy = a^2 - b$,

$\therefore$ $x - y = \pm\sqrt{a^2 - b}$ and $x + y = a$,

$\therefore$ $x = \dfrac{a \pm \sqrt{a^2 - b}}{2}$ and $y = \dfrac{a \mp \sqrt{a^2 - b}}{2}$.

$$\therefore \quad \sqrt{a + \sqrt{b}} = \sqrt{\frac{a \pm \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a \mp \sqrt{a^2 - b}}{2}}.$$

If $a^2 - b$ is a perfect square, $\sqrt{a^2 - b}$ is rational and it is seen that $\sqrt{a + \sqrt{b}}$ can be expressed as the sum of two

quadratic surd forms, one of which, however, may be rational.

Thus, in $11 + 4\sqrt{7}$ or $11 + \sqrt{112}$, $a = 11$, $b = 112$, and $a^2 - b = 121 - 112 = 9$, which is a square. Therefore the square root of $11 + 4\sqrt{7}$ can be found, as shown in the preceding example.

57. Since $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2 = x + y + z + 2\sqrt{xy} + 2\sqrt{xz} + 2\sqrt{yz}$, it follows that if an expression of the form $a + \sqrt{b} + \sqrt{c} + \sqrt{d}$ is a square, the square root may be of the form $\sqrt{x} + \sqrt{y} + \sqrt{z}$.

Ex.—Find the square root of $12 + 4\sqrt{3} + 4\sqrt{5} + 2\sqrt{15}$.

Let $\sqrt{(12 + 4\sqrt{3} + 4\sqrt{5} + 2\sqrt{15})} = \sqrt{x} + \sqrt{y} + \sqrt{z}$.

Squaring $12 + 4\sqrt{3} + 4\sqrt{5} + 2\sqrt{15} = x + y + z + 2\sqrt{xy} + 2\sqrt{yz} + 2\sqrt{xz}$.

$\therefore\ x + y + z = 12,\ 2\sqrt{xy} = 4\sqrt{3},\ 2\sqrt{yz} = 4\sqrt{5},\ 2\sqrt{xz} = 2\sqrt{15}$.

Solving $xy = 12,\ yz = 20,\ xz = 15$,

we obtain $x = 3,\ y = 4,\ z = 5$.

Since these values satisfy the equation $x + y + z = 12$, therefore the square root $= \sqrt{3} + \sqrt{4} + \sqrt{5} = 2 + \sqrt{3} + \sqrt{5}$.

Verify this result by squaring.

If the values found for x, y, z had not satisfied the equation $x + y + z = 12$, we could conclude that the given expression had not a square root of the form $\sqrt{x} + \sqrt{y} + \sqrt{z}$.

EXERCISE 20

Find the square root of :

1. $33 - 18\sqrt{2}$. **2.** $101 + 28\sqrt{13}$. **3.** $2\frac{1}{3}\frac{1}{6} - \frac{1}{3}\sqrt{2}$.

4. $2a + 2\sqrt{a^2 - b^2}$. **5.** $(a + b)^2 - 4(a - b)\sqrt{ab}$.

6. $2a - \sqrt{3a^2 - 2ab - b^2}$. **7.** $1 + a^2 + \sqrt{1 + a^2 + a^4}$.

8. Find the fourth root of $56 + 24\sqrt{5}$ and of $284 - 48\sqrt{35}$.

9. Simplify $\sqrt{6 - \sqrt{17 + 12\sqrt{2}}}$ and $\sqrt{\dfrac{6 + 2\sqrt{3}}{33 - 19\sqrt{3}}}$.

10. If $x = \frac{\sqrt{3}}{2}$, find the value of $\frac{1+x}{1+\sqrt{1+x}} + \frac{1-x}{1-\sqrt{1-x}}$.

11. If $\sqrt{a+\sqrt{b}} = x + \sqrt{y}$, show that $\sqrt{a-\sqrt{b}} = x - \sqrt{y}$.

12. If $\sqrt[3]{a-\sqrt{b}} = x - \sqrt{y}$, show that $\sqrt[3]{a+\sqrt{b}} = x + \sqrt{y}$.

13. If $a + \sqrt{b}$ is a root of $x^2 + px + q = 0$, where a, p and q are rational, show that $a - \sqrt{b}$ is also a root.

14. Solve $x^4 - 39x^2 + 66x - 28 = 0$, being given that one root is $3 + \sqrt{5}$.

15. If $\sqrt[3]{26 - 15\sqrt{3}} = x - \sqrt{y}$, then $\sqrt[3]{26 + 15\sqrt{3}} = x + \sqrt{y}$. Hence show that $x^2 - y = 1$ and $x^3 + 3xy = 26$. Solve by trial for x and y and thus find the cube root of $26 - 15\sqrt{3}$.

16. Find the cube root of $72 - 32\sqrt{5}$ and of $99 + 70\sqrt{2}$.

17. Find the value of $\sqrt[2]{35 + 8\sqrt{6}} - \sqrt[3]{10 + 6\sqrt{3}}$ to three decimal places.

18. Find the square root of $6 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}$ and of $22 - 12\sqrt{2} + 6\sqrt{5} - 4\sqrt{10}$.

19. If $x = \frac{1}{2}(\sqrt{5} + 1)$, show that

$$\frac{1}{\sqrt{x} - \sqrt{2-x}} - \frac{1}{\sqrt{x} + \sqrt{2-x}} = \pm 1.$$

58. Roots of Unity.

If x is a square root of unity,

then, $$x^2 = 1 \quad \text{or} \quad x = \pm 1.$$

It is thus seen that unity has two real square roots, which differ only in sign.

If x is a cube root of unity,

then, $$x^3 = 1 \quad \text{or} \quad x^3 - 1 = 0,$$

$$\therefore \ (x-1)(x^2 + x + 1) = 0,$$

$$\therefore \ x = 1 \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}.$$

It is thus seen that unity has three cube roots, only one of which is real.

Similarly, the four fourth roots of unity are the roots of $x^4 - 1 = 0$ or of $(x - 1)(x + 1)(x^2 + 1) = 0$.

$$\therefore \quad x = \pm 1 \quad \text{or} \quad \pm \sqrt{-1}.$$

Therefore unity has four fourth roots, two of which are real.

Also the fifth roots of unity are the roots of

$$x^5 - 1 = 0 \quad \text{or} \quad (x - 1)(x^4 + x^3 + x^2 + x + 1) = 0,$$

$$\therefore \quad x = 1 \quad \text{or} \quad x^4 + x^3 + x^2 + x + 1 = 0.$$

Since $x^4 + x^3 + x^2 + x + 1 = 0$ is a reciprocal equation, it may be solved as in Art. 7, Ex. 3.

$$x^4 + x^3 + x^2 + x + 1 = 0,$$

$$\therefore \quad x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} = 0,$$

$$\therefore \quad \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0,$$

$$\therefore \quad \left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 1 = 0,$$

$$\therefore \quad x + \frac{1}{x} = \frac{-1 \pm \sqrt{5}}{2}.$$

Let $\dfrac{-1 \pm \sqrt{5}}{2} = a$, then $x + \dfrac{1}{x} = a$,

$$\therefore \quad x^2 - ax + 1 = 0,$$

$$\therefore \quad x = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

If the value of a be substituted, each of these four roots will be found to be imaginary.

Therefore unity has five fifth roots, only one of which is real.

Similarly, by solving $x^6 - 1 = 0$, we will find that unity has six sixth roots, two of which are real.

59. The Cube Roots of Unity. We have found that unity has three cube roots, which are

$$1, \quad \frac{-1 + \sqrt{-3}}{2}, \quad \frac{-1 - \sqrt{-3}}{2}.$$

The two imaginary cube roots were found by solving the equation $x^2 + x + 1 = 0$.

If we denote these imaginary cube roots by m and n, then since m and n are cube roots of unity

$$m^3 = n^3 = 1.$$

Also since m and n are roots of $x^2 + x + 1 = 0$.

$$\begin{aligned} mn &= 1. \\ \therefore\ m^3n &= m^2, \text{ on multiplying by } m^2, \\ \therefore\ n &= m^2, \text{ since } m^3 = 1, \\ \text{Similarly, } m &= n^2. \end{aligned}$$

Therefore each imaginary cube root is the square of the other. Show that this is true by squaring each of them.

If we represent either of the imaginary cube roots by ω, then the other would be ω^2, and the three cube roots would be $1, \omega, \omega^2$.

Since ω is a root of $x^2 + x + 1 = 0$, then $\omega^2 + \omega + 1 = 0$, or $\omega^2 + \omega = -1$, $\omega^2 + 1 = -\omega$, $\omega + 1 = -\omega^2$.

Ex. 1.—Every three consecutive powers of ω give the three cube roots of unity.

Thus,
$$\begin{aligned} \omega^{18} &= (\omega^3)^6 &&= 1, \\ \omega^{19} &= \omega^{18} \cdot \omega &&= \omega, \\ \omega^{20} &= \omega^{18} \cdot \omega^2 &&= \omega^2. \end{aligned}$$

Also,
$$\begin{aligned} \omega^{3n} &= (\omega^3)^n &&= 1, \\ \omega^{3n+1} &= \omega^{3n} \cdot \omega &&= \omega, \\ \omega^{3n+2} &= \omega^{3n} \cdot \omega^2 &&= \omega^2. \end{aligned}$$

Ex. 2.—Find the three cube roots of 27.

The arithmetical cube root is evidently 3.

Since $(3\omega)^3 = 27\omega^3 = 27$, since $\omega^3 = 1$, it is seen that 3ω is also a cube root of 27. Similarly, $3\omega^2$ is another cube root.

Therefore the three cube roots of 27 are $3, 3\omega, 3\omega^2$.

Similarly, it follows that if a is the arithmetical cube root of any number, then $a\omega$ and $a\omega^2$ are the two imaginary cube roots of the same number.

Ex. 3.—Express $x^3 + y^3$ as the product of three linear factors.

$$x^3 + y^3 = x^3 + (\omega y)^3 = x^3 + (\omega^2 y)^3.$$

From the form $x^3 + y^3$ it follows that $x + y$ is a factor. From the form $x^3 + (\omega y)^3$, $x + \omega y$ is a factor, and from $x^3 + (\omega^2 y)^3$, $x + \omega^2 y$ is a factor.

The factors $x + y$, $x + \omega y$, $x + \omega^2 y$ are all different, and since the expression is of the third degree

$$\therefore \quad x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y).$$

Since $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$,

it follows that $x^2 - xy + y^2 = (x + \omega y)(x + \omega^2 y)$.

That this is true may be verified by multiplication,

for $(x + \omega y)(x + \omega^2 y) = x^2 + xy(\omega + \omega^2) + \omega^3 y^2 = x^2 - xy + y^2$,

since $\omega^2 + \omega = -1$.

Ex. 4.—Factor $a^3 + b^3 + c^3 - 3abc$.

$a + b + c$ is known to be one factor

Since $a^3 + b^3 + c^3 - 3abc = a^3 + (\omega b)^3 + (\omega^2 c)^3 - 3a \cdot \omega b \cdot \omega^2 c$, it is seen that $a + \omega b + \omega^2 c$ is also a factor, and similarly $a + \omega^2 b + \omega c$ is a factor,

$\therefore \quad a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$.

Show by multiplication that

$$a^2 + b^2 + c^2 - ab - bc - ca = (a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c).$$

EXERCISE 21

1. What are the three cube roots of 125? Of $\frac{1}{8}a^3$?

2. What are the linear factors of $x^3 - y^3$, $a^3 + \frac{8}{b^3}$, $x^2 + xy + y^2$, $x^3 - y^3 - z^3 - 3xyz$?

Show that

3. $(1 + \omega)^4 = \omega^2$. 4. $(1 + \omega^2)^4 = \omega$ 5. $(1 + \omega - \omega^2)^3 = -8$.

6. $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$.

7. $(1 + \omega)^3 = (1 + \omega^2)^3 = -1$.

8. $(1 + \omega - \omega^2)^2 + (1 - \omega + \omega^2)^2 + (1 - \omega - \omega^2)^2 = 0$.

9. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$.

10. $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \ldots$ to $2n$ factors $= 2^{2n}$.

11. $(a + b)^2 + (a\omega + b\omega^2)^2 + (a\omega^2 + b\omega)^2 = 6ab$.

12. $(a + b)^3 + (a\omega + b\omega^2)^3 + (a\omega^2 + b\omega)^3 = 3a^3 + 3b^3$.

13. Show that $(1 + \omega)^2$ is a cube root of 1, and that $(2 + \omega)^2$ is a cube root of -27.

14. Simplify $\dfrac{1}{a + b + c} + \dfrac{1}{a + b\omega + c\omega^2} + \dfrac{1}{a + b\omega^2 + c\omega}$.

15. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, show that

(1) $x^2 + y^2 + z^2 = 6ab$. (2) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$.

(3) $x^4 + y^4 + z^4 = 18a^2b^2$. (4) $x^5 + y^5 + z^5 = 15ab(a^3 + b^3)$.

(5) $xyz = -(x + y)(y + z)(z + x) = a^3 + b^3$.

CHAPTER VIII

THEORY OF EQUATIONS

THE simpler parts of the theory of quadratic equations are found in elementary texts. For the purpose of review this part of the theory is given in a concise form.

60. Sum and Product of the Roots. Every quadratic equation, in one unknown, may be reduced to the standard form

$$ax^2 + bx + c = 0.$$

The roots are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Representing these roots by m and n, we have by addition and multiplication

$$m + n = -\frac{b}{a}, \text{ and } mn = \frac{c}{a}.$$

If the roots are equal in magnitude but opposite in sign, then $m + n = 0$ or $-\dfrac{b}{a} = 0$ or $b = 0$.

If the roots are reciprocals, their product is unity, and therefore $\dfrac{c}{a} = 1$ or $c = a$.

61. Factors of $ax^2 + bx + c$. If m and n are the roots of the corresponding equation, then

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right), \\ &= a\{x^2 - (m + n)x + mn\}, \\ &= a(x - m)(x - n). \end{aligned}$$

Therefore the factors of $ax^2 + bx + c$ are $a(x - m)(x - n)$, where m and n are the roots of $ax^2 + bx + c = 0$.

62. A Quadratic Equation cannot have more than two Roots. We have seen that $ax^2 + bx + c = 0$ has two roots. If we represent these roots by m and n we have shown that

$$ax^2 + bx + c = a(x - m)(x - n).$$

If $a(x - m)(x - n) = 0$, then $a = 0$, $x - m = 0$ or $x - n = 0$.

But a cannot be zero, as the equation would not then be a quadratic,

$$\therefore \quad x - m = 0 \quad \text{or} \quad x - n = 0.$$

But no values of x other than m and n will make $x - m$ or $x - n$ equal to zero.

Therefore, m and n are the only roots.

Since the equation $ax^2 + bx + c = 0$ has only two roots, then the quadratic expression $ax^2 + bx + c$ can be resolved into linear factors in only one way.

63. Character of the Roots. If we examine the roots of the standard quadratic equation $ax^2 + bx + c = 0$, we see that the following conclusions may be drawn:

(1) If the particular values of a, b, c are such that $b^2 - 4ac = 0$, the roots are equal, for each is $-\frac{b}{2a}$.

(2) If $b^2 - 4ac$ is a perfect square, then its square root can be found exactly and the roots are rational.

(3) If $b^2 - 4ac$ is not a perfect square, but is positive, the roots are real and irrational.

(4) If $b^2 - 4ac$ is negative the roots are imaginary.

The expression $b^2 - 4ac$ is called the **discriminant of the** equation $ax^2 + bx + c = 0$.

64. Character of the Factors of $ax^2 + bx + c$. If the roots of the corresponding equation are equal, then $ax^2 + bx + c$ has two linear factors in x which are equal, and therefore

$ax^2 + bx + c$ is a perfect square, with respect to x, when $b^2 - 4ac = 0$.

If $b^2 - 4ac$ is a perfect square, the expression $ax^2 + bx + c$ has two rational factors, for under this condition the corresponding equation has rational roots.

Ex. 1.—If $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ has equal roots, find m and verify the result.

Here the discriminant $= 4(1 + 3m)^2 - 28(3 + 2m)$. If the roots are equal $(1 + 3m)^2 - 7(3 + 2m) = 0$,

$$\therefore \quad 9m^2 - 8m - 20 = 0,$$

$$\therefore \quad m = 2 \text{ or } -\frac{10}{9}.$$

If $m = 2$, the given equation becomes $x^2 - 14x + 49 = 0$, which is seen to have equal roots, since $14^2 = 4 \cdot 49$.

If $m = -\frac{10}{9}$, the equation becomes $9x^2 + 42x + 49 = 0$, which has equal roots, since $42^2 = 4 \cdot 9 \cdot 49$.

Ex. 2.—Prove that the roots of $(x - a)(x - b) = h^2$ are always real, if a, b and h are real quantities.

The equation is $x^2 - x(a + b) + ab - h^2 = 0$.

The discriminant $= (a + b)^2 - 4(ab - h^2)$,

$= (a - b)^2 + 4h^2$.

Since the square of every real quantity is positive, it follows that $(a - b)^2$ and $4h^2$ are positive.

Therefore the discriminant is positive and the roots are real.

Ex. 3.—Factor $221x^2 + 8x - 165$.

If we use the formula to solve $221x^2 + 8x - 165 = 0$, we find the roots to be $\frac{11}{13}$ and $-\frac{15}{17}$.

$$\therefore \quad 221x^2 + 8x - 165 = 221\left(x - \frac{11}{13}\right)\left(x + \frac{15}{17}\right)$$

$$= (13x - 11)(17x + 15).$$

Ex. 4.—If p and q are the roots of $ax^2 + bx + c = 0$, find the values of $p^2 + q^2$, $p^3 + q^3$, $p^4 + q^4$.

$$p^2 + q^2 = (p + q)^2 - 2pq = \left(-\frac{b}{a}\right)^2 - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}.$$

$$p^3 + q^3 = (p + q)^3 - 3pq(p + q) = \left(-\frac{b}{a}\right)^3 - 3\frac{c}{a}\left(-\frac{b}{a}\right) = \frac{3abc - b^3}{a^3}.$$

$$p^4 + q^4 = (p^2 + q^2)^2 - 2p^2q^2 = \text{etc.}$$

Ex. 5.—If m and n are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $m + h$ and $n + h$.

The sum of the roots of the required equation

$$= m + n + 2h$$
$$= -\frac{b}{a} + 2h.$$

The product

$$= (m + h)(n + h) = mn + h(m + n) + h^2 = \frac{c}{a} - \frac{bh}{a} + h^2.$$

Therefore the equation is

$x^2 - x$(sum of the roots) + (product of roots) $= 0$,

or $$x^2 - x\left(-\frac{b}{a} + 2h\right) + \left(\frac{c}{a} - \frac{bh}{a} + h^2\right) = 0,$$

or $$ax^2 + x(b - 2ah) + c - bh + ah^2 = 0.$$

Ex. 6.—Find the condition if one root of $px^2 + qx + r = 0$ is n times the other root.

Let one root be a, then the other is na.

$$\therefore\ a + na = -\frac{q}{p},\quad \therefore\ a = \frac{-q}{p(1 + n)},\quad \therefore\ na = \frac{-nq}{p(1 + n)}.$$

The product of the roots $= \frac{r}{p}$. $\therefore\ \frac{nq^2}{p^2(1 + n)^2} = \frac{r}{p}$.

$\therefore\ nq^2 = pr(1 + n)^2$, which is the required condition.

EXERCISE 22

1. Without solving, determine the character of the roots of:

(*a*) $3x^2 - 5x + 1 = 0$. (*b*) $3x^2 - 5x + 2 = 0$.

(*c*) $3x^2 - 5x + 3 = 0$. (*d*) $3x^2 - 5x + 2\frac{1}{12} = 0$.

2. For what values of m will the following equations have equal roots: (*a*) $x^2 - mx + 36 = 0$, (*b*) $6x^2 + 3x + m = 0$, (*c*) $mx^2 - 28x + 49 = 0$, (*d*) $(m + 2)x^2 - 20mx + 30(2m + 3) = 0$.

3. Factor

(*a*) $20x^2 - 93x + 108$, (*b*) $1081x^2 - 24x - 1$,
(*c*) $(2a^2 - 3ab + b^2)x^2 + a^2x - 6a^2 + 5ab - b^2$.

4. If m and n are the roots of $3x^2 - 6x + 5 = 0$, find the values of $m^2 + n^2$, $\frac{m}{n} + \frac{n}{m}$, $m^3 + n^3$, $m^4 + 6m^2n^2 + n^4$.

5. Find the equation whose roots are m and n where $m + n = 10$ and $mn = 16$; where $m + n = 4$, $m^2 + n^2 = 20$.

6. Prove that if one of the equations $x^2 + x(3c - b) + bc = 0$, $x^2 - x(5c - b) + 4c^2 = 0$ has equal roots, so has the other.

7. Prove that $3x^2 - 5x - 1$ and $2x^2 - 3x - 4$ cannot be equal for any real value of x.

8. If one root of $2x^2 + 5x = a(5x + 2)$ is $-\frac{3}{4}$, find the other root.

9. Prove that the roots of $x^2 + bx + c = 0$ are rational if $b = k + \frac{c}{k}$, where b, c, k are rational.

10. Prove that $x^2 - 2(a + 2)x - a^2 = 0$ cannot have equal roots for any real value of a.

11. If $(m - 1)x^2 - (4m + 4)x + 7m + 1 = 0$ has equal roots, find m.

12. Show that $8a^2x(2x - 1) + b^2 = 0$ and $4a^2x^2 + b^2(4x + 1) = 0$ cannot both have real roots unless $a = \pm b$.

13. Show that the roots of $x^2(q^2 - pr) + 2x(p + r) = 4$ are always real, for real values of p, q, r.

14. If m and n are the roots of $x^2 + px + q = 0$, find the values of $m^3 + n^3$ and $m^4 + n^4$.

15. If a and b are the roots of $x^2 - px + q = 0$, find the equation whose roots are $a + 2b$ and $b + 2a$.

16. If the roots of $ax^2 + bx + c = 0$ are x_1 and x_2, find the value of (1) $(ax_1 + b)(ax_2 + b)$, (2) $(ax_1 + b)^{-2} + (ax_2 + b)^{-2}$.

17. Find the condition that one root of $ax^2 + bx + c = 0$ may be five times the other root.

18. If m and n are the roots of $x^2 + px + q = 0$, find the roots of $x^2 + 4px + 16q = 0$.

19. If $4x^2 - 6xy + 21y^2 + a(2x^2 - 3xy - 2y^2)$ is a perfect square, find the values of a and verify.

20. Form the equation whose roots are the cubes of the roots of $x^2 - ax + b = 0$.

21. Form the quadratic equation of which the sum of the roots is a and the sum of the cubes of the roots is b^3.

22. If one root of $x^2 - x(2a - 3) + a^2 - 3a - 4 = 0$ is three times the other, find a.

23. One root of $x^2 - (k + 1)x + 2k + 1 = 0$ exceeds the other by 2. Find the values of k and verify.

24. If a and b are the roots of $px^2 + qx + r = 0$, find the equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$.

25. Find, in its simplest form, the equation whose roots are

$$\frac{a}{1 + \sqrt{1 - ab}} \text{ and } \frac{a}{1 - \sqrt{1 - ab}}.$$

26. If the difference between the roots of $ax^2 + bx + c = 0$ is equal to the difference between the roots of $px^2 + qx + r = 0$, show that $p^2(b^2 - 4ac) = a^2(q^2 - 4pr)$.

27. Find the quadratic function of x which vanishes when $x = 3 - \sqrt{5}$ and when $x = 3 + \sqrt{5}$ and which equals -4 when $x = 5$.

28. Find the equation whose roots are the reciprocals of the geometric and harmonic means between the roots of $x^2 - 10x + 16 = 0$.

29. Find the equation whose roots are the square of the sum and the square of the difference of the roots of

$$2x^2 + 2x(a + b) + a^2 + b^2 = 0.$$

30. If the roots of $c(x^2 + m^2) + cmx + dm^2x^2 = 0$ are p and q, prove that $c(p^2 + q^2) + cpq + dp^2q^2 = 0$.

31. One root of $ax^2 + bx + c = 0$ is double the other, and one root of $ax^2 + cx + b = 0$ is four times the other. Prove that $b^3 : c^3 = 18 : 25$.

32. Show that $m(x^2 - y^2) - xy(l - n)$ always admits of two real linear factors, if l, m, n are real.

33. If the equations $ax + by = 1$, $cx^2 + dy^2 = 1$ have a pair of equal roots, show that $\frac{a^2}{c} + \frac{b^2}{d} = 1$ and that $x = \frac{a}{c}$, $y = \frac{b}{d}$.

34. Show that, if the roots of $ax^2 + bx + c = 0$ are real, they are both positive if b differs in sign from a and c. What is the condition

that both roots will be negative; also that they will differ in sign, the negative root being the greater numerically?

65. Variations in the Sign of $ax^2 + bx + c$ for real Values of x. Consider the expression $x^2 - 11x + 28$.

Since $x^2 - 11x + 28 = (x - 4)(x - 7)$, it follows that

(1) the expression is zero where $x = 4$ or 7,

(2) it is negative for any value of x between 4 and 7, for then the two factors differ in sign,

(3) it is positive for all values of x which are greater than 7 or less than 4. For if x is greater than 7 both factors are positive, and if x is less than 4 both factors are negative.

The roots of the equation $x^2 - 11x + 28 = 0$ are 4 and 7, and it is seen that the expression $x^2 - 11x + 28$ is negative only for values of x which lie between the roots of the corresponding equation.

Consider the expression $x^2 - 6x + 20$.

Since $x^2 - 6x + 20 = (x - 3)^2 + 11$, it is seen that this expression is positive for all real values of x.

Here the roots of $x^2 - 6x + 20 = 0$ are imaginary, since $6^2 - 4 \cdot 20$ is negative.

These are particular cases of the following general theorem.

66. The Expression $ax^2 + bx + c$ has the same Sign as a for all real Values of x, except when the Roots of $ax^2 + bx + c = 0$ are real and unequal and x lies between them.

Let the roots of $ax^2 + bx + c = 0$ be m and n, then

$$ax^2 + bx + c = a(x - m)(x - n).$$

(1) Let the roots be real and equal, then since $m = n$

$$ax^2 + bx + c = a(x - n)^2.$$

But $(x - n)^2$ is positive for all real values of x,

$\therefore$ $ax^2 + bx + c$ has the same sign as a.

(2) Let the roots be real and unequal, and let m be greater than n.

Then if x is greater than m both $x - m$ and $x - n$ are positive, and therefore $(x - m)(x - n)$ is positive.

If x is less than n both $x - m$ and $x - n$ are negative, and therefore $(x - m)(x - n)$ is positive.

$\therefore$ if x is greater than m or less than n, $(x - m)(x - n)$ is positive, and $ax^2 + bx + c$ has the same sign as a.

But if x lies between m and n, $x - n$ is positive and $x - m$ is negative and $(x - m)(x - n)$ is negative.

$\therefore$ if x lies between m and n the expression $ax^2 + bx + c$ differs in sign from a.

(3) Let the roots be imaginary so that $b^2 - 4ac$ is negative.

Then $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}.$$

But $\left(x + \frac{b}{2a}\right)^2$ is positive for real values of x and $\frac{4ac - b^2}{4a^2}$ is positive, since b^2 is less than $4ac$,

$\therefore$ $ax^2 + bx + c$ has the same sign as a when the roots of $ax^2 + bx + c = 0$ are imaginary.

It is thus seen that $ax^2 + bx + c$ differs in sign from a only in the case in which the roots of the corresponding equation are real and unequal, and x has a value lying between them.

Ex. 1.—Show that (*a*) $3x^2 + 7x + 5$ is always positive, and (*b*) $6x - x^2 - 10$ is always negative for real values of x.

(*a*) The equation $3x^2 + 7x + 5 = 0$ has imaginary roots, since $7^2 - 4 \cdot 3 \cdot 5$ is negative.

$\therefore$ $3x^2 + 7x + 5$ has the same sign as the coefficient of x^2 and is therefore always positive.

(*b*) The roots of $6x - x^2 - 10 = 0$ or $x^2 - 6x + 10 = 0$ are also imaginary and therefore the sign of $6x - x^2 - 10$ is the same as the coefficient of x^2 and is always negative.

It might be more satisfactory to treat such particular cases independently of the general theorem.

Thus in (*b*), $6x - x^2 - 10 = -(x^2 - 6x + 10)$,
$= -\{(x - 3)^2 + 1\}$.

But $(x - 3)^2 + 1$ is positive for real values of x and therefore $6x - x^2 - 10$ is always negative.

Ex. 2.—For what values of x is $6x^2 - x - 15$ positive, and for what values is it negative?

The roots of $6x^2 - x - 15 = 0$ are $\frac{5}{3}$ and $-\frac{3}{2}$, therefore $6x^2 - x - 15$ is negative for values of x between $\frac{5}{3}$ and $-\frac{3}{2}$, but is positive for all other real values of x.

Similarly, the expression $-6x^2 + x + 15$ is positive for all values of x between $\frac{5}{3}$ and $-\frac{3}{2}$ and is negative for all other real values of x.

EXERCISE 23

1. The following are negative for what real values of x:
(1) $2x^2 - 11x + 14$. (2) $6x^2 - 29x - 22$. (3) $x^2 - 4x - 1$.

2. The following are positive for what real values of x:
(1) $21 + 4x - x^2$. (2) $2x^2 - 11x + 15$. (3) $6x - x^2 - 4$.

3. Show that these are positive for all real values of x:
(1) $x^2 - 5x + 10$. (2) $5x(x - 3) + 13$. (3) $\frac{1}{3}x^2 - \frac{1}{2}x + \frac{1}{5}$.

4. For what values of x is $65x^2 - 241x + 86$ equal to zero? For what values is it negative and for what values positive?

5. If the roots of $x^2 + x(a - 4) + 9 = 0$ are real, show that a cannot have any value between 10 and -2.

6. Show that $x^2 + px + q$ is always positive for real values of x, if p^2 is less than $4q$.

7. Find the only integral value of x for which the expression $6x^2 - 61x + 154$ is negative.

8. For what integral values of x is the expression $55x - 6x^2 - 121$ positive?

9. Show that a quadratic expression in x will have the same sign for all real values of x if its discriminant is negative or zero.

10. For what real values of x is $27 - 10x + x^2$ less than $12x - 29 - x^2$?

67. We have seen that the expression $ax^2 + bx + c$ is a perfect square if $b^2 - 4ac = 0$, and that it has two rational factors if $b^2 - 4ac$ is a perfect square.

Here $ax^2 + bx + c$ is the general form of an expression of the second degree in one variable x.

The general form of an expression of the second degree in two variables x and y is

$$ax^2 + bxy + cy^2 + dx + ey + f.$$

Writing it in descending powers of x, we have

$$ax^2 + x(by + d) + (cy^2 + ey + f).$$

This expression has two rational factors

if $(by + d)^2 - 4a(cy^2 + ey + f)$ is a perfect square,

or if $y^2(b^2 - 4ac) + y(2bd - 4ae) + (d^2 - 4af)$ is a square.

But this last expression is a perfect square if

$$(2bd - 4ae)^2 - 4(b^2 - 4ac)(d^2 - 4af) = 0.$$

This is therefore the condition that the general expression of the second degree in x and y may be the product of two rational linear factors.

When simplified the condition becomes

$$ae^2 + cd^2 + fb^2 = bde + 4acf.$$

Ex.—If $6x^2 + xy - 2y^2 + 5x + 8y + k$ is the product of two rational linear factors, find k.

Arranging as a quadratic expression in x, we have

$$6x^2 + x(y + 5) - (2y^2 - 8y - k).$$

Since this expression has two rational factors

$(y + 5)^2 + 24(2y^2 - 8y - k)$ is a perfect square,

or $49y^2 - 182y + 25 - 24k$ is a perfect square.

$\therefore \quad 182^2 - 196(25 - 24k) = 0.$

Solving this equation we find $k = -6$.

Verify this result by finding the factors of

$$6x^2 + xy - 2y^2 + 5x + 8y - 6.$$

68. Ex. 1.—Find the conditions that the equations

$$ax^2 + bx + c = 0 \text{ and } px^2 + qx + r = 0$$

may have the same roots.

Suppose m and n are the roots of each equation.

Then $m + n = -\frac{b}{a}$ and $m + n = -\frac{q}{p}$, $\therefore \frac{a}{p} = \frac{b}{q}$.

Also $mn = \frac{c}{a}$ and $mn = \frac{r}{p}$, $\therefore \frac{a}{p} = \frac{c}{r}$.

$$\therefore \frac{a}{p} = \frac{b}{q} = \frac{c}{r}.$$

It is thus seen that when two quadratic equations have the same roots, they are virtually the same equation, the one being obtained from the other by multiplying by a common factor.

Ex. 2.—Find the condition that the equations

$$ax^2 + bx + c = 0 \quad \text{and} \quad px^2 + qx + r = 0$$

may have one common root.

Suppose that m is a root of each

$$\therefore \quad am^2 + bm + c = 0,$$

and $$pm^2 + qm + r = 0.$$

$$\therefore \quad \frac{m^2}{br - cq} = \frac{m}{cp - ar} = \frac{1}{aq - bp}. \qquad \text{(Art. 41.)}$$

$$\therefore \quad m^2 = \frac{br - cq}{aq - bp} \text{ and } m = \frac{cp - ar}{aq - bp}.$$

$$\therefore \quad \frac{br - cq}{aq - bp} = \frac{(cp - ar)^2}{(aq - bp)^2}.$$

$\therefore (aq - bp)(br - cq) = (cp - ar)^2$, which is the condition required.

This is also seen to be the condition that the two expressions $ax^2 + bx + c$ and $px^2 + qx + r$ may have a common linear factor.

69. The Remainder Theorem. If $ax^2 + bx + c$ is divided by $x - m$ the remainder is $am^2 + bm + c$.

Here it is seen that the remainder might be obtained by substituting m for x in the given dividend, or the remainder is the same function of m that $ax^2 + bx + c$ is of x.

$$\begin{array}{r|l} x - m & ax^2 + bx + c \;(ax + (b + am) \\ & \underline{ax^2 - amx} \\ & (b + am)x + c \\ & \underline{(b + am)x - m(b + am)} \\ & c + bm + am^2 \end{array}$$

If we represent $ax^2 + bx + c$ by the symbol $f(x)$, then the remainder $am^2 + bm + c$ would be $f(m)$.

That is, in this case, when $f(x)$ is divided by $x - m$ the remainder is $f(m)$.

We will show that this theorem is always true.

Let $f(x)$ be any rational integral function of x.

Suppose $f(x)$ is divided by $x - m$ until the remainder does not contain x. Let the quotient be represented by Q and the remainder by R.

Then $$f(x) = Q(x - m) + R.$$

Since this equation is true for all values of x, we may substitute m for x. If we suppose that Q then becomes Q_1, we have

$$f(m) = Q_1(m - m) + R = R.$$

$\therefore$ **when $f(x)$ is divided by $x - m$ the remainder is $f(m)$.**

This is called the **remainder theorem.** The following deductions from it are important.

(1) If $f(m) = 0$, then $f(x)$ is exactly divisible by $x - m$.

(2) If m is a root of the equation $f(x) = 0$, then $x - m$ is a factor of $f(x)$.

Ex. **1.**—Find the remainder on dividing

$$x^4 - 10x^3 + 3x^2 + 13x + 50 \text{ by } x - 2.$$

Here $$f(2) = 2^4 - 10.2^3 + 3.2^2 + 13.2 + 50$$
$$= 16 - 80 + 12 + 26 + 50 = 24.$$

Therefore the remainder is 24. Verify by division.

Ex. **2.**—If, a, b, c are the roots of $x^3 + px^2 + qx + r = 0$, find the relations connecting the roots and the coefficients.

Since a, b, c are roots of the equation, three factors of the expression $x^3 + px^2 + qx + r$ are $x - a$, $x - b$, $x - c$.

But the expression is of the third degree and can have only three linear factors in x,

$$\therefore \quad x^3 + px^2 + qx + r = (x - a)(x - b)(x - c),$$
$$= x^3 - x^2(a + b + c) + x(ab + bc + ca) - abc.$$

Equating the coefficients of like powers of x the required relations are

$a + b + c = -p$ = the sum of the roots.
$ab + bc + ca = q$ = the sum of the products two at a time.
$abc = -r$ = the product of the roots.

Ex. 3.—If p, q, r are the roots of $x^3 + ax^2 + bx + c = 0$, find the values of $p^2 + q^2 + r^2$ and $p^3 + q^3 + r^3$.

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp) = (-a)^2 - 2b = a^2 - 2b.$$
$$\begin{aligned} p^3 + q^3 + r^3 &= (p^3 + q^3 + r^3 - 3pqr) + 3pqr \\ &= (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) + 3pqr \\ &= -a(a^2 - 2b - b) - 3c \\ &= -a^3 + 3ab - 3c. \end{aligned}$$

Ex. 4.—Show that $a - b$, $b - c$, $c - a$ are factors of

$$a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3)$$

and find the remaining factor.

If we substitute a for b the expression becomes zero and therefore $a - b$ is a factor. Since the expression is symmetrical $b - c$ and $c - a$ are also factors.

Since the given expression is of the fifth degree, the remaining factor must be of the second degree and must be symmetrical with regard to a, b and c.

Therefore the remaining factor must be of the form

$$m(a^2 + b^2 + c^2) + n(ab + bc + ca).$$

Then $a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3)$
$= (a - b)(b - c)(c - a)\{m(a^2 + b^2 + c^2) + n(ab + bc + ca)\}$.

Since this is an identity we may substitute any values for a, b, c which will not make either side zero.

If $a = 1$, $b = 2$, $c = 0$, we have $4 = 10m + 4n$.
If $a = -1$, $b = 1$, $c = 0$, we have $2 = -4m + 2n$.
Solving, we get $m = 0$, $n = 1$.

$$\therefore\ a^2(b^3 - c^3) + b^2(c^3 - a^3) + c^2(a^3 - b^3) = (a - b)(b - c)(c - a)(ab + bc + ca).$$

70. Zero and Infinite Roots of $ax^2 + bx + c = 0$.

Let m and n be the roots, then $m + n = -\frac{b}{a}$ and $mn = \frac{c}{a}$.

(1) If one root is zero, then the product is zero, and

$$\therefore\ \frac{c}{a} = 0 \quad \text{or} \quad c = 0.$$

The equation then becomes $ax^2 + bx = 0$, and the roots are seen to be $-\frac{b}{a}$ and 0.

(2) If both roots are zero, then the sum and the product of the roots are zero.

$$\therefore \quad -\frac{b}{a} = 0 \quad \text{or} \quad b = 0 \quad \text{and} \quad \frac{c}{a} = 0 \quad \text{or} \quad c = 0.$$

$\therefore$ if $b = c = 0$, both roots are zero and the equation becomes $ax^2 = 0$.

(3) To find when the roots are infinite, let $x = \frac{1}{y}$.

Since $$ax^2 + bx + c = 0,$$

$$\therefore \quad \frac{a}{y^2} + \frac{b}{y} + c = 0 \quad \text{or} \quad cy^2 + by + a = 0.$$

One value of y will be zero if $a = 0$, and both values of y will be zero if $a = 0$ and $b = 0$.

But if $y = 0$, then $\frac{1}{x} = 0$ or $x = \infty$.

$\therefore$ $ax^2 + bx + c = 0$ has one infinite root if $a = 0$, and both roots are infinite if $a = b = 0$.

EXERCISE 24

1. If $x^2 + 2xy - 3x + my - 10$ has two rational linear factors, find m and verify the results.

2. If $x^2 + mxy - 6y^2 + 5x + 6$ is the product of two rational linear factors, find the values of m.

3. Find the condition that $x^2 + ax + b = 0$ and $x^2 + px + q = 0$ may have a common root.

4. If $x^2 + 6x + b$ and $x^2 + 12x + 3b$ have a common linear factor, what numerical values can b have and what is the common factor in each case?

5. If $x^2 + ax + 10$ and $x^2 + bx - 10$ have a common factor, show that $a^2 - b^2 = 40$.

6. If the equations $x^2 - 7x + c = 0$ and $x^2 - 9x + 2c = 0$ have a common root, solve both of them.

7. If $ax^2 + bx + c = 0$ and $a^2x^2 + b^2x + c^2 = 0$ have a common root, find that root.

8. The equations $x^2 - 2ax + b = 0$ and $x^2 - 2bx + a = 0$ have a common root if $4a + 4b + 1 = 0$.

9. Show that $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ will have a common root if $a - b = 0$ or $a + b + 1 = 0$.

10. What value of m will make $4x^4 - 12x^3 + mx^2 - 6x + 1$ divisible by $2x - 1$? Verify the result.

11. When $x^4 + px^3 + 5x^2 + qx + 6$ is divided by $x - 2$ the remainder is 16, and when divided by $x + 1$ the remainder is 10. Find p and q.

12. Find all the factors of $(a + b + c)^4 - (b + c)^4 - (c + a)^4 - (a + b)^4 + a^4 + b^4 + c^4$, being given that a is one of them.

13. Show that $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$
$= (a - b)(b - c)(c - a)(a + b + c)$.

14. Show that $(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2)$.

15. Show that $a - b$, $b - c$, $c - a$, are factors of $(a - b)^5 + (b - c)^5 + (c - a)^5$ and find the remaining factor.

16. Show that $(a + b)(b + c)(c + a)$ is a factor of $(a + b + c)^5 - a^5 - b^5 - c^5$ and find the remaining factor.

17. Find the values of a and b so that $2x^2 - 41x + b = 0$ and $6x^3 - 137x^2 + ax - 91 = 0$ may have two roots in common.

18. Find the sum of the squares and the sum of the cubes of the roots of $x^3 - 4x^2 + 3x - 4 = 0$.

19. Solve $2x^3 - 5x^2 - 4x + 12 = 0$, it being given that two of the roots are equal.

20. Find the equation whose roots are the reciprocals of the roots of $ax^3 + bx^2 + cx + d = 0$.

21. Find the equation whose roots are each greater by unity than the roots of $x^3 - 5x^2 + 7x - 11 = 0$. (Let $y = x + 1$ and substitute $y - 1$ for x.)

22. If $a + b + c = 9$, $ab + bc + ca = 26$, $abc = 24$, show that a, b, c are the roots of $x^3 - 9x^2 + 26x - 24 = 0$. What are the values of a, b, c?

23. Solve $x + y + z = 12$, $xy + yz + zx = 47$, $xyz = 60$.

24. Solve $x + y + z = 0$, $x^2 + y^2 + z^2 = 14$, $xyz = 6$.

25. Solve $x^2 + y^2 + z^2 = 14$, $xy + yz + zx = 1$, $xyz = -6$.

26. Solve $x + y + z = 1$, $x^2 + y^2 + z^2 = 49$, $x^3 + y^3 + z^3 = 181$.

27. If $x + y + z = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ and $xyz = 1$, show that x, y, z are the cube roots of unity.

28. Find the value of $x^5 - 16x^4 - 20x^3 + 46x^2 + 83x + 35$ when $x = 17$.

29. If $f(x)$ be any rational integral function of x, show that $f(x) - f(a)$ is divisible by $x - a$.

30. Find the remainder when $f(x)$ is divided by $(x - a)(x - b)$.

31. Find the condition that $axy + bx + cy + d$ may be the product of two factors of the form $mx + n$, $py + q$.

32. Find the value of a if $3x^2 - 10xy + ay^2 - 2x - 2y - 21$ is the product of two rational linear factors in x and y.

33. If $3x^2 + 2mxy + 2y^2 + 2ax - 4y + 1$ can be resolved into two rational linear factors, then $m^2 + 4am + 2a^2 + 6 = 0$.

34. If $6x^2 - xy - 12y^2 - 17x + 12 = 0$, show that for every real value of x there is a real value of y, and for every real value of y there is a real value of x.

CHAPTER IX

INEQUALITIES, MAXIMA AND MINIMA

71. Inequality. The statement that one quantity is unequal to another is called an **inequality.**

Thus, if a, b, c denote the lengths of the sides of a triangle we know that $a + b$ is greater than c, or that $a + b - c$ is a positive quantity. These statements are usually written

$$a + b > c \quad \text{or} \quad a + b - c > 0.$$

72. Quantities restricted in Value. Quantities like a, $a + b$, $a - b$, ab, may have any value whatever, depending upon the values of the letters involved.

This is not true, however, concerning all algebraic quantities.

Thus, the quantity x^2, where x is real, must be positive, whether x is positive or negative.

We can therefore conclude that the least value (or minimum value) of every square quantity is zero.

Thus, $(x - y)^2$ being a square, must be positive,

$$\therefore \quad x^2 - 2xy + y^2 > 0, \text{ unless } x = y,$$
$$\therefore \quad x^2 + y^2 > 2xy.$$

That is, the sum of the squares of any two unequal real quantities is greater than twice their product.

This corresponds to the geometrical theorem :

" The sum of the squares on any two unequal straight lines is greater than twice the rectangle contained by the lines."

73. If the Sum of any Number of Squares is Zero, then each must be Zero.

Thus, if $x^2 + y^2 + z^2 = 0$, then $x = y = z = 0$.

For if x, y, z represent real quantities, no one of the quantities x^2, y^2, z^2 can be negative, therefore their sum cannot be zero unless each one is zero.

Ex. 1.—What are the only real values of x and y which satisfy the equation $(x - a)^2 + (y - b)^2 = 0$?

It is at once seen that $x - a$ and $y - b$ must equal zero.
Therefore the only real values are $x = a$ and $y = b$.

We should distinguish clearly between the conclusion

when $(x - a)^2 + (y - b)^2 = 0$, ... (1)

and when $(x - a)(y - b) = 0$. . . . . (2)

From (1) it follows that $x = a$ *and* $y = b$.

From (2) it follows that $x = a$ *or* $y = b$.

Ex. 2.—If $a^2 + b^2 + c^2 = ab + bc + ca$, then $a = b = c$.

Since $a^2 + b^2 + c^2 - ab - bc - ca = 0$,

$\therefore\ 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$,

$\therefore\ (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$,

$\therefore\ a - b = 0,\ b - c = 0,\ c - a = 0$,

$\therefore\ a = b = c$.

74. Maximum and Minimum Values. Since the square of a real quantity is always positive, it follows that:

(1) The minimum value of a quantity like $x^2 + 4$ is 4 for all real values of x.

Similarly, the minimum of $(x - 3)^2 + 9$ is 9 and of $5(x - 2)^2 - 11$ is -11.

(2) The maximum value of $3 - x^2$ is 3 and of $5 - 2(x + 3)^2$ is 5.

Ex. 1.—For all real values of x find the minimum value of $x^2 + 4x + 11$.

Express the literal part of $x^2 + 4x + 11$ as a square.

Then $$x^2 + 4x + 11 = x^2 + 4x + 4 + 7$$
$$= (x + 2)^2 + 7.$$

Therefore the minimum value of $x^2 + 4x + 11$ is 7, and it has this minimum value when $x = -2$.

We might also have proceeded thus:

Let $$x^2 + 4x + 11 = y,$$

$\therefore$ $$x^2 + 4x + 11 - y = 0,$$

$\therefore$ $$x = \frac{-4 \pm \sqrt{16 - 4(11 - y)}}{2} = -2 \pm \sqrt{y - 7}.$$

Now, since x is real, $y - 7$ must be positive.

$\therefore$ y cannot be less than 7.

Ex. 2.—Find the minimum value of $3x^2 - 6x + 40$.

Let $$3x^2 - 6x + 40 = y,$$

$\therefore$ $$3x^2 - 6x + 40 - y = 0.$$

Since x is real, $b^2 - 4ac$ must be positive,

$\therefore$ $36 - 12(40 - y)$ or $12y - 444$ is positive.

$\therefore$ the minimum value of y is 37.

Ex. 3.—Find the maximum value of $5 - 7x - 6x^2$.

Let $$5 - 7x - 6x^2 = y,$$

$\therefore$ $$6x^2 + 7x + y - 5 = 0.$$

Since x is real, $49 - 24(y - 5)$ or $169 - 24y$ is positive,

$\therefore$ the maximum value of y, or $5 - 7x - 6x^2$, is $\frac{169}{24}$.

Ex. 4.—The expression $ax^2 + bx + c$ has a minimum value, for real values of x, when a is positive and a maximum value when a is negative.

For $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$
$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

Since $\left(x + \frac{b}{2a}\right)^2$ is positive, $\frac{4ac - b^2}{4a}$ is the minimum value of $ax^2 + bx + c$ when a is positive and is the maximum value when a is negative.

EXERCISE 25

1. In what respects are the following quantities restricted for real values of x:

$$x^2 + 5,\ 5 - x^2,\ 3(x-1)^2 - 7,\ 7 - 3(x-1)^2\ ?$$

2. What conclusions can be drawn from the following:

$$x^2 + y^2 = 0,\ 2a^2 + 3b^2 = 0,\ (x-1)^2 + (y-2)^2 = 0,$$
$$(x-1)(y-2) = 0\ ?$$

3. Find the minimum values of $x^2 - 2x + 15$, $2x^2 - 8x + 9$ $5x^2 - 3x + \frac{1}{2}$, for real values of x.

4. For real values of x find the maximum values of $3 - 4(x-1)^2$, $(x-1)(2-x)$, $(4-x)^2 - (3-x)^2 - (2-x)^2$.

5. Find for what real value of x the expression $2x^2 + 11x - 21$ takes its minimum value.

6. Prove that $(a+b)^2 > 4ab$ unless a and b are equal. What geometrical proposition corresponds to this?

7. Since $a^2 + b^2 > 2ab$, etc., deduce that

$$a^2 + b^2 + c^2 > ab + bc + ca$$

except when $a = b = c$. State the conclusion geometrically.

8. If x is real, prove that $x^2 - 6x + 21$ can never be less than 12, and that $3 + 4x - x^2$ can never be more than 7.

9. For what real value of x has $x(20-x)$ its greatest value? What is the maximum value?

10. A line whose length is a is divided into two parts x and $a - x$. Show that $x(a-x)$ is greatest when $x = \frac{1}{2}a$. Hence show that the rectangle contained by the segments of a line is greatest when the line is bisected.

11. Show that $\dfrac{a+b}{2} > \sqrt{ab} > \dfrac{2ab}{a+b}$, if a and b are real, unequal and positive.

12. Show that $(a+b)(a^3+b^3) > (a^2+b^2)^2$, if a and b have like signs and are unequal.

13. Show that for positive values of x, $x + \dfrac{1}{x} > 2$ unless $x = 1$. Hence show that the sum of any positive number and its reciprocal is never less than 2.

14. Since $(x+y)^2 - (x-y)^2 = 4xy$, show that if the sum of two positive quantities is constant their product is a maximum when the

quantities are equal, and if their product is constant their sum is a minimum when they are equal.

15. The perimeter of a rectangle is 80 in. Find its greatest area.

16. Find the least perimeter of a rectangle whose area is 256 sq. in.

17. What are the only real values of x and y which will satisfy the equation $x^2 - 6x + 9 + y^2 + 4y + 4 = 0$?

18. Solve $x^2 + y^2 - 8x - 6y + 25 = 0$, for real values of x and y.

19. If $a^2 + c^2 + 2b^2 = 2b(a + c)$, show that $a = b = c$.

20. Since $a + b > 2\sqrt{ab}$, etc., prove $(a + b)(b + c)(c + a) > 8abc$.

21. If $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$, show that $x : a = y : b = z : c$.

22. Show that $x^2 + px + q$ has its minimum value, for real values of x, when x equals one-half the sum of the roots of $x^2 + px + q = 0$.

23. If a is positive, for what real value of x will $ax^2 + bx + c$ have its minimum value?

24. Show that the maximum value of $(x - a)(b - x)$ is $\frac{1}{4}(a - b)^2$.

25. Show that the roots of $\dfrac{1}{x} + \dfrac{1}{x + a} + \dfrac{1}{x + b} = 0$ are always real, if a and b are real.

26. If $a^3 + b^3 + c^3 = 3abc$, show that $a + b + c = 0$ or $a = b = c$.

27. If $a^2 + b^2$ is constant show that ab is greatest when $a = b$. Hence show that when the hypotenuse of a right-angled triangle is constant, the area is a maximum when the triangle is isosceles.

28. If $(a + b)^2 + (b + c)^2 + (c + d)^2 = 4(ab + bc + cd)$, show that $a = b = c = d$.

29. Show that $(x - a)^2 + (y - b)^2 + (a^2 + b^2 - 1)(x^2 + y^2 - 1) = 0$ is equivalent to $(ax + by - 1)^2 + (bx - ay)^2 = 0$ and find the only real values of x and y that satisfy it.

30. For real values of x show that $x^3 + 1 > = \text{or} < x^2 + x$ according as $x > \mp$ or < -1. except when $x = 1$.

31. If a, b, c are positive, show that $a^3 + b^3 > ab(a + b)$ and deduce that $2(a^3 + b^3 + c^3) > bc(b + c) + ca(c + a) + ab(a + b)$.

32. Divide the odd integer $2n + 1$ into two integral parts whose product is a maximum.

33. If $x < 1$, show that $7x^2 + 11 > x^3 + 17x$.

34. Show that $a^4 + b^4 > ab(a^2 + b^2)$ unless $a = b$.

35. If a and b are real and $a + b\sqrt{-1} = 0$, show that $a = b = 0$.

36. If x and y are positive and $x + y = 1$, show that the least value of $\frac{1}{x} + \frac{1}{y}$ is 4.

75. Sometimes an algebraic quantity has both a maximum and a minimum value for real values of the variable.

Ex. 1.—Determine the limits of the value of $\frac{x^2 - x + 1}{x^2 + x + 1}$ for real values of x.

Let $$\frac{x^2 - x + 1}{x^2 + x + 1} = y,$$

then $$x^2(1 - y) - x(1 + y) + 1 - y = 0.$$

Since the values of x are real, $b^2 - 4ac$ is positive.

$\therefore$ $(1 + y)^2 - 4(1 - y)^2$ is positive,
$\therefore$ $-3y^2 + 10y - 3$ is positive,
$\therefore$ $3y^2 - 10y + 3$ is negative,
$\therefore$ $(y - 3)(3y - 1)$ is negative.

$\therefore$ y must lie between 3 and $\frac{1}{3}$. (Art. 66.)

Therefore the maximum value of $\frac{x^2 - x + 1}{x^2 + x + 1}$ is 3 and the minimum value is $\frac{1}{3}$.

Ex. 2.—Determine the limits of $\frac{x^2 - 24}{2x - 11}$ for real values of x.

Let $$\frac{x^2 - 24}{2x - 11} = y, \qquad \therefore \quad x^2 - 2xy + 11y - 24 = 0.$$

Since x is real, $4y^2 - 4(11y - 24)$ is positive,
or $4y^2 - 44y + 96$ is positive,
or $4(y - 8)(y - 3)$ is positive.

$\therefore$ y cannot have a value between 8 and 3.

That is, $\frac{x^2 - 24}{2x - 11}$ can have any real value whatever except those which lie between 8 and 3.

Ex. 3.—Show that, for real values of x, $\frac{x^2 - 4x + 3}{x^2 - 6x + 8}$ can have any real value.

Let $\frac{x^2 - 4x + 3}{x^2 - 6x + 8} = y, \quad \therefore \quad x^2(1 - y) - x(4 - 6y) + 3 - 8y = 0.$

Since x is real, $(4 - 6y)^2 - 4(1 - y)(3 - 8y)$ is positive,

or $4y^2 - 4y + 4$ is positive,

or $y^2 - y + 1$ is positive.

Since the roots of $y^2 - y + 1 = 0$ are imaginary, the expression $y^2 - y + 1$ is positive for all real values of y. (Art. 66.)

Therefore $\frac{x^2 - 4x + 3}{x^2 - 6x + 8}$ can have any real value for real values of x.

EXERCISE 26

Determine the limits of the values of the following functions for real values of x:

1. $\frac{x^2 - 4x - 20}{x - 7}$. **2.** $\frac{x^2 + x + 1}{x^2 + 1}$. **3.** $\frac{x^2 - 3x + 3}{x - 1}$.

4. $\frac{x + 1}{x^2 + 3}$. **5.** $\frac{x^2 - 5x + 1}{x^2 - x + 1}$. **6.** $\frac{x^2 - bc}{2x - b - c}$.

7. For real values of x, find the maximum and minimum values of

$$\frac{x^2 - 3x + 4}{x^2 + 3x + 4}.$$

8. Show that the maximum value of $\frac{1}{5 + 2x + x^2}$ is $\frac{1}{4}$.

9. Show that $\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17}$ is a maximum when $x = 1$ and is a minimum when $x = 2$.

10. If $1 - 2x - 7x^2 = y(1 + x + x^2)$ and x is real, find the maximum value of y.

11. Show that $\frac{x^2 - 20x + 400}{x - 20}$ can have no positive value less than 60.

12. If x is real, show that $\frac{x^2 + ab}{2x - a + b}$ cannot have a value between a and $-b$.

13. Show that $\frac{4x^2 + 2x - 1}{x^2 + 6x + 3}$ may have any real value if x is real.

14. Show that $\frac{2x^2 + 3x + 3}{x^2 - 2x + 5}$ is positive for real values of x.

15. In selling an article the loss per cent. is equal to the cost in dollars. Show that the selling price cannot exceed \$25.

16. If a and b are positive numbers whose product is 54, find the values of a and b for which $\frac{a}{4} + \frac{b}{6}$ is least.

17. The hypotenuse of a right-angled triangle is 20 in. Show that the maximum area is 100 sq. in.

18. If $9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$ and x and y are real, then x must lie between 3 and 6, and y between 1 and 10.

19. For real values of x, show that the maximum value of $x + \sqrt{24x - x^2 - 112}$ is 20 and the minimum value is 4.

CHAPTER X

MISCELLANEOUS SERIES

76. In this chapter will be found the summation of certain series which do not belong to the types already considered, but some of them will be seen to be combinations of those types.

If we multiply together the corresponding terms of the A.P.,

$$1 + 2 + 3 + 4 + \ldots\ldots + n$$

and the G.P.,

$$1 + r + r^2 + r^3 + \ldots\ldots + r^{n-1},$$

we get

$$1 + 2r + 3r^2 + 4r^3 + \ldots + nr^{n-1}.$$

This series is sometimes called an **arithmetico-geometric series,** as it is a combination of the terms of an A.P. and a G.P. It must be recognized that it is not the *complete product* of these two series.

Ex. 1.—Sum $1 + 2r + 3r^2 + \ldots + nr^{n-1}$.

Let $S = 1 + 2r + 3r^2 + 4r^3 + \ldots + nr^{n-1}$,

$\therefore \quad Sr = \quad r + 2r^2 + 3r^3 + \ldots + (n-1)r^{n-1} + nr^n$,

$\therefore \quad S(1-r) = 1 + r + r^2 + r^3 + \ldots\ldots + r^{n-1} - nr^n$,

$$= \frac{1-r^n}{1-r} - nr^n, \qquad \left[S = a \cdot \frac{1-r^n}{1-r}\right]$$

$$\therefore \quad S = \frac{1-r^n}{(1-r)^2} - \frac{nr^n}{1-r}.$$

If $r < 1$, when n is indefinitely large r^n is indefinitely small. The sum to infinity may therefore be found by substituting 0 for r^n in the sum to n terms.

$\therefore$ the sum to infinity $= \frac{1}{(1-r)^2}$, when $r < 1$.

Check this result by dividing 1 by $1 - 2r + r^2$.

The sum to infinity, when $r < 1$, may be found independently of the sum to n terms.

Let $S = 1 + 2r + 3r^2 + 4r^3 + \ldots$ to infinity,

$\therefore \quad Sr = \quad r + 2r^2 + 3r^3 + \ldots\ldots$ to infinity,

$\therefore \quad S(1-r) = 1 + r + r^2 + r^3 + \ldots\ldots$ to infinity,

$$= \frac{1}{1-r}, \qquad \left[S = \frac{a}{1-r}\right]$$

$$\therefore \quad S = \frac{1}{(1-r)^2}.$$

Ex. 2.—Sum to n terms and to infinity when $r < 1$,

$$a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \ldots\ldots$$

This series is the general form of the series in Ex. 1.

Let $S = a + (a+d)r + (a+2d)r^2 + \ldots + (a + \overline{n-1}d)r^{n-1}$

$\therefore \quad Sr = \quad ar + (a+d)r^2 + \ldots + (a + \overline{n-2}d)r^{n-1}$
$\qquad + (a + \overline{n-1}d)r^n.$

$\therefore \quad S(1-r) = a + dr + dr^2 + dr^3 + \ldots + dr^{n-1} - (a + \overline{n-1}d)r^n$

$= a + dr(1 + r + r^2 + \ldots\ldots$ to $n-1$ terms$)$
$\qquad - (a + \overline{n-1}d)r^n$

$$= a + dr \cdot \frac{1 - r^{n-1}}{1-r} - (a + \overline{n-1}d)r^n.$$

$$\therefore \quad S = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{(a + \overline{n-1}d)r^n}{1-r}.$$

If n is indefinitely large and $r < 1$, r^n and r^{n-1} each equals 0,

$\therefore$ the sum to infinity $= \frac{a}{1-r} + \frac{dr}{(1-r)^2}$.

Ex. 3.—Sum to n terms and to infinity the series

$$1 + (a+b)x + (a^2+ab+b^2)x^2 + (a^3+a^2b+ab^2+b^3)x^3 + \ldots$$

Multiply each side by $a - b$,

$$\therefore \quad (a-b)S = (a-b) + (a^2-b^2)x + (a^3-b^3)x^2 + \dots$$
$$= (a + a^2x + a^3x^2 + \dots) - (b + b^2x + b^3x^2 + \dots)$$
$$= a \cdot \frac{1-a^nx^n}{1-ax} - b \cdot \frac{1-b^nx^n}{1-bx}$$
$$\therefore \quad S = \frac{a(1-a^nx^n)}{(a-b)(1-ax)} - \frac{b(1-b^nx^n)}{(a-b)(1-bx)}.$$

If ax and bx are each numerically less than unity the sum to infinity

$$= \frac{a}{(a-b)(1-ax)} - \frac{b}{(a-b)(1-bx)} = \frac{1}{(1-ax)(1-bx)}.$$

EXERCISE 27

Find the sum of:

1. $1 + 3x + 5x^2 + 7x^3 + \dots$ to infinity, where $x < 1$.

2. $1 + 3a + 5a^2 + 7a^3 + \dots$ to n terms.

3. $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \dots$ to n terms and to infinity.

4. $1 + \frac{3}{4} + \frac{5}{16} + \frac{7}{64} + \dots$ to n terms and to infinity.

5. $r + 2r^2 + 3r^3 + \dots$ to n terms.

6. $1 + 3x + 7x^2 + 15x^3 + 31x^4 + \dots$ to infinity, where $x < 1$.

7. $1 + (1+x)r + (1+x+x^2)r^2 + (1+x+x^2+x^3)r^3 + \dots$ to n terms. Under what conditions can it be summed to infinity? What is then the sum to infinity?

8. Find the sum of n terms of the series whose r^{th} term is $(2r+1)x^r$.

9. If $x = 1 + \frac{1}{n}$, show that n^2 is the sum of n terms of the series $1 + 2x + 3x^2 + 4x^3 + \dots$

10. Sum $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$ to infinity, where $x < 1$.

11. Sum $1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8} + \dots$ to n terms and to infinity.

12. Show that $2^{\frac{1}{2}} \cdot 4^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot 16^{\frac{1}{16}} \dots$ to infinity $= 4$.

13. Find to the nearest dollar the sum of 20 terms of $\$10 + \frac{\$20}{1{\cdot}05} + \frac{\$30}{1{\cdot}05^2} + \frac{\$40}{1{\cdot}05^3} + \dots$ if $1{\cdot}05^{-19} = {\cdot}39573$, and $1{\cdot}05^{-20} = {\cdot}37689$. What is the sum to infinity?

77. Sums of Powers of the Natural Numbers. We have already found, as an A.P., the sum of the first n natural numbers. Denote the sum by S_1.

Then $$S_1 = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}.$$

To find the sum of the squares of the first n natural numbers. Denote the sum by S_2.

Then $$S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2.$$

We have

$$x^3 - (x-1)^3 = x^3 - (x^3 - 3x^2 + 3x - 1) = 3x^2 - 3x + 1.$$

Since this is an identity and therefore true for all values of x, we will give x in succession the values n, $n-1$, $n-2 \ldots 3, 2, 1$.

$$\therefore \quad n^3 - (n-1)^3 = 3n^2 - 3n + 1$$
$$(n-1)^3 - (n-2)^3 = 3(n-1)^2 - 3(n-1) + 1$$
$$(n-2)^3 - (n-3)^3 = 3(n-2)^2 - 3(n-2) + 1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$3^3 - 2^3 = 3(3)^2 - 3(3) + 1$$
$$2^3 - 1^3 = 3(2)^2 - 3(2) + 1$$
$$1^3 - 0^3 = 3(1)^2 - 3(1) + 1$$

$\therefore$ by addition $$n^3 = 3S_2 - 3S_1 + n$$

$$\therefore \quad 3S_2 = n^3 + 3S_1 - n$$
$$= n^3 - n + \frac{3n(n+1)}{2}$$
$$= n(n+1)(n-1) + \frac{3n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left\{2(n-1) + 3\right\}$$
$$= \frac{n(n+1)(2n+1)}{2}$$

$$\therefore \quad S_2 = \frac{n(n+1)(2n+1)}{6}.$$

Check this result by putting $n = 1, 2, 3 \ldots$

Ex. 1.—Sum to n terms $1 \cdot 3 + 3 \cdot 7 + 5 \cdot 11 + \ldots$

The nth term consists of two factors, the first of which is the nth term of the A.P. 1, 3, 5, . . . which is $2n - 1$, and the second is the nth term of the A.P. 3, 7, 11, . . . which is $4n - 1$.

$\therefore$ the nth term $= (2n - 1)(4n - 1) = 8n^2 - 6n + 1$.

Since the nth term $= 8n^2 - 6n + 1$

the $(n - 1)$th term $= 8(n - 1)^2 - 6(n - 1) + 1$

the $(n - 2)$th term $= 8(n - 2)^2 - 6(n - 2) + 1$

. . . . . .

the 2nd term $= 8(2)^2 - 6(2) + 1$

the 1st term $= 8(1)^2 - 6(1) + 1$

$$\therefore \quad S = 8S_2 - 6S_1 + n$$

$$= 8 \cdot \frac{n(n+1)(2n+1)}{6} - 6 \cdot \frac{n(n+1)}{2} + n$$

$$= \frac{n}{3}(8n^2 + 3n - 2).$$

Check by substituting particular values for n.

Ex. 2.—Sum $1^2 + 4^2 + 7^2 + \ldots$ to n terms.

The nth term $= (3n - 2)^2 = 9n^2 - 12n + 4$.

Proceed as in Ex. 1, and the sum $= 9S_2 - 12S_1 + 4n$

$$= \frac{n}{2}(6n^2 - 3n - 1).$$

78. To find the Sum of the Cubes of the first n Natural Numbers. Denote the sum by S_3.

Then $S_3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$.

Since $x^4 - (x - 1)^4 = 4x^3 - 6x^2 + 4x - 1$

$\therefore$ $n^4 - (n - 1)^4 = 4n^3 - 6n^2 + 4n - 1$

$(n - 1)^4 - (n - 2)^4 = 4(n - 1)^3 - 6(n - 1)^2 + 4(n - 1) - 1$

. . . . . .

$2^4 - 1^4 = 4(2)^3 - 6(2)^2 + 4(2) - 1$

$1^4 - 0^4 = 4(1)^3 - 6(1)^2 + 4(1) - 1$

$\therefore$ by addition, $n^4 = 4S_3 - 6S_2 + 4S_1 - n$

$$\therefore \quad 4S_3 = n^4 + 6S_2 - 4S_1 + n$$

$$= n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$= n^4 + 2n^3 + n^2 = n^2(n+1)^2$$

$$\therefore \quad S_3 = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

It will be seen that $S_3 = S_1^2$, that is, the sum of the cubes of the first n natural numbers is equal to the square of their sum.

It would be well to remember that

$$S_1 = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

$$S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$S_3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Ex.—Sum $1.3^2 + 2.5^2 + 3.7^2 + \ldots$ to n terms.

The nth term $= n(2n+1)^2 = 4n^3 + 4n^2 + n$.

$\therefore$ the sum of n terms $= 4S_3 + 4S_2 + S_1$

$$= n^2(n+1)^2 + \frac{2n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)(6n^2+14n+7)}{6}.$$

EXERCISE 28

1. Find the sum of the squares and the sum of the cubes of the first 20 natural numbers.

2. Find the sum of the squares of all integers with two digits.

3. Sum $1.2 + 2.3 + 3.4 + \ldots$ to n terms.

4. Sum $1^2 + 3^2 + 5^2 + 7^2 + \ldots$ to n terms.

5. Sum $11^2 + 13^2 + 15^2 + \ldots + 97^2 + 99^2$.

6. Sum $1.2.3 + 2.3.4 + 3.4.5 + \ldots$ to n terms.

7. Sum $1.2^2 + 4.4^2 + 7.6^2 + 10.8^2 + \ldots$ to n terms.

8. Sum to n terms the series whose nth term is

(1) $3n^2 - 2n$, (2) $n(n^2 - 5)$, (3) $n^3 - \frac{3}{2}n$.

9. The nth term of a series is $n(2n - 18)$. For how many terms would the sum be zero?

10. The sum of n terms of a series is $n^3 + 3n^2 - n$. Find the nth term.

11. Since the sum of the cubes of the first n natural numbers is a perfect square, show that the cube of any integer is the difference of two squares. Find two squares whose difference is 1728.

12. Sum $1^3 + 3^3 + 5^3 + \ldots$ to n terms by expressing it in the form ($1^3 + 2^3 + 3^3 + \ldots$ to $2n$ terms) $-$ ($2^3 + 4^3 + 6^3 + \ldots$ to n terms).

13. Sum $(a + b)^2 + (a + 2b)^2 + (a + 3b)^2 + \ldots + (a + nb)^2$.

14. Sum $n + 2(n - 1) + 3(n - 2) + \ldots$ to n terms.

15. Show that the sum of the cubes of n terms of the series whose nth term is $n - 2$ is $\frac{1}{4}(n - 2)^2(n - 1)^2 - 1$.

16. Find the sum of n terms of the series whose nth term is

$$(a + \overline{n - 1}b)^2.$$

17. Sum to n terms the series whose nth term is $4n^3 - 2^n$.

18. If $S_a = 1 + 3 + 5 + \ldots$ to n terms and $S_b = 1^3 + 3^3 + 5^3 + \ldots$ to n terms, show that $S_b = 2S_a^2 - S_a$.

19. Sum $1^2 - 2^2 + 3^2 - 4^2 + \ldots$ (1) to $2n$ terms, (2) to $2n + 1$ terms.

20. Sum $1 + 2^2 + 3 + 4^2 + 5 + 6^2 + \ldots$ to n terms, (1) when n is even, (2) when n is odd.

21. Sum $1^4 + 2^4 + 3^4 + 4^4 + \ldots$ to n terms.

22. Show that the sum of the cubes of any number of consecutive integers is divisible by the sum of those integers.

79. Mathematical Induction. The way in which mathematical induction is used in proving the truth of certain algebraic theorems is illustrated in the following examples.

Ex. 1.—Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n + 1)^2}{4}$.

(1) Suppose the theorem is true for the sum of r terms. Then the sum of $r + 1$ terms = sum of r terms + the $(r + 1)$th term

$$= \frac{r^2(r + 1)^2}{4} + (r + 1)^3$$

$$= (r + 1)^2\left(\frac{r^2}{4} + r + 1\right) = \frac{(r + 1)^2(r + 2)^2}{4}.$$

But this sum is the same function of $r + 1$ that $\dfrac{r^2(r + 1)^2}{4}$ is of r.

Therefore if the theorem is true for the sum of r terms, it is also true for $r + 1$ terms.

(2) If we put $n = 1$, we see that the theorem is true, since

$$1^3 = \frac{1^2(1+1)^2}{4}.$$

(3) We have shown that if the theorem is true for any value of n it is also true for the next higher value.

But it is true when $n = 1$, therefore it is true when $n = 2$, therefore when $n = 3$, and so on.

Therefore it is true for all values of n.

In this method of proof there are three distinct steps:

(1) We prove that if we assume the theorem true for any value of n, it must be true for the next higher value.

(2) We then examine if the theorem is true for the smallest value of n, in this case when $n = 1$.

(3) We then deduce that it is true when $n = 2$, and therefore when $n = 3$ and so on for all values of n.

Ex. 2.—If n is a positive integer show that

$$n(n+1)(2n+1) \text{ is divisible by } 6.$$

Let $n(n+1)(2n+1) = f(n)$ and suppose the theorem true when $n = r$.

Let $f(r) = r(r+1)(2r+1) = 6x$, where x is an integer.

$$\therefore \quad f(r+1) - f(r) = (r+1)(r+2)(2r+3) - r(r+1)(2r+1)$$
$$= (r+1)\{(2r^2+7r+6) - (2r^2+r)\}$$
$$= (r+1)(6r+6) = 6(r+1)^2.$$

$$\therefore \quad f(r+1) = f(r) + 6(r+1)^2$$
$$= 6x + 6(r+1)^2.$$

$\therefore$ $f(r+1)$ is divisible by 6, if $f(r)$ is divisible by 6.

But $f(1) = 1.2.3$, which is divisible by 6, $\therefore$ $f(2)$ is divisible, $\therefore$ $f(3)$ is divisible, and so on.

$\therefore$ $f(n)$ is divisible by 6 for all integral values of n.

Ex. 3.—Prove that $x^n - a^n$ is divisible by $x + a$ when n is any even integer.

Assume that $x^r - a^r$ is divisible by $x + a$, then we wish to prove that $x^{r+2} - a^{r+2}$ is also divisible.

Dividing $x^{r+2} - a^{r+2}$ by $x + a$ to two terms we get

$$\frac{x^{r+2} - a^{r+2}}{x + a} = x^{r+1} - x^r a + \frac{x^r a^2 - a^{r+2}}{x + a}.$$

$$= x^{r+1} - x^r a + \frac{a^2(x^r - a^r)}{x + a}.$$

$\therefore$ if $x^r - a^r$ is divisible by $x + a$, $x^{r+2} - a^{r+2}$ is also divisible.

But $x^n - a^n$ is divisible by $x + a$ when $n = 2$, therefore it is divisible when $n = 4$, when $n = 6$, and so on.

$\therefore$ $x^n - a^n$ is divisible by $x + a$ for all even values of n.

EXERCISE 29

Prove by mathematical induction :

1. $1 + 3 + 5 + 7 + \ldots\ldots$ to n terms $= n^2$.

2. $1.2 + 2.3 + 3.4 + \ldots\ldots$ to n terms $= \frac{1}{3}n(n+1)(n+2)$.

3. $a + ar + ar^2 + \ldots\ldots$ to n terms $= a \cdot \frac{r^n - 1}{r - 1}$.

4. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots$ to n terms $= \frac{n}{n+1}$.

5. $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \ldots\ldots$ to n terms $= \frac{n}{3n+1}$.

6. $\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots\ldots$ to n terms $= \frac{2n}{n+1}$.

7. $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots\ldots$ to n terms $= \frac{n(n+3)}{4(n+1)(n+2)}$.

8. $x^n - a^n$ is divisible by $x - a$ when n is any positive integer and $x^n + a^n$ is divisible by $x + a$ when n is any odd positive integer.

9. When n is a positive integer $n^2 + 3n$ is divisible by 2.

10. If n is an even integer $n^2 + 2n$ is divisible by 8.

11. When n is a positive integer $(3n + 1)7^n - 1$ is divisible by 9 and $9^n - 8n - 1$ is divisible by 64.

12. If a_1, a_2, a_3, $\ldots$ a_n are n consecutive terms of an A.P., show that $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}$.

13. Show that $n^3 + 20n$ is divisible by 48 if n is an even integer.

14. The sum of $(1^5 + 1^7) + (2^5 + 2^7) + (3^5 + 3^7) + \ldots$ to n terms

$$= \frac{n^4(n+1)^4}{8}.$$

CHAPTER XI

PERMUTATIONS AND COMBINATIONS

80. THE character of the problems to be discussed in this chapter will be illustrated by the following simple examples.

Ex. **1.**—Write down all the numbers with two different digits, which can be formed from the digits 1, 2, 3, 4. How many are there?

The numbers are: 12, 13, 14; 21, 23, 24; 31, 32, 34; 41, 42, 43.
There are therefore 12 numbers. Note that there are 3 beginning with 1, 3 with 2, 3 with 3, and 3 with 4.
The total number is 4 times 3 or 12.

Ex. **2.**—In how many ways can a vowel and a consonant be selected from the 5 vowels and 21 consonants of the alphabet, the vowel being chosen first?

If the vowel a is chosen, the selections are:

$$ab,\ ac,\ ad,\ af,\ ag,\ \ldots\ldots\ldots ay,\ az.$$

Similarly, if the vowel e is chosen they are:

$$eb,\ ec,\ ed,\ ef,\ eg,\ \ldots\ldots\ldots ey,\ ez.$$

How many are there beginning with a? How many beginning with e? With i? With o? With u?
The total number is 5 times 21 or 105.

Ex. **3.**—From a committee of 5 persons, a chairman and a secretary are to be chosen. In how many ways can this be done?

Suppose the 5 persons are represented by A, B, C, D, E.
If A is chosen as chairman, then B, C, D or E may be secretary. If B is chairman then A, C, D or E may be secretary, and so on.

All the ways in which the two may be chosen are :

AB	*BA*	*CA*	*DA*	*EA*
AC	*BC*	*CB*	*DB*	*EB*
AD	*BD*	*CD*	*DC*	*EC*
AE	*BE*	*CE*	*DE*	*ED*

Here there are 5 ways in which the chairman may be chosen, and after he is chosen there are 4 ways in which the secretary may be chosen and the total number of ways in which the two may be chosen is 4×5 or 20.

If a treasurer was to be chosen from the remaining three, there would be 3 choices for the treasurer. That is, with each of the 20 ways of choosing a chairman and secretary there would be 3 ways of completing the list of officers.

Therefore the total number of ways of choosing the three officers is $5 \times 4 \times 3$ or 60.

It will be seen that the solutions of these problems all depend upon the same fundamental principle.

81. Fundamental Theorem. If one operation can be performed in a different ways and after it is performed in any one way another operation can be performed in b ways, the two operations can be performed together in ab different ways.

Thus in Ex. 1, there were four ways of writing down the first digit; it might be 1, 2, 3 or 4. After the first digit was written down there were three ways of writing down the second digit; it might be any one of the three remaining digits. Therefore the total number of ways of writing down the *two* digits was 4×3 or 12.

The theorem might be extended to any number of operations.

Ex.—In how many ways can all the letters of the word *absurd* be written down?

Here there are 6 choices in writing down the first letter, since any one of the 6 letters may be chosen.

When the first is written down there are 5 letters left and there are therefore 5 choices for the second letter.

Therefore the first two letters can be written in 6×5 ways.

When the first two letters are written there are 4 choices for the third letter and therefore the first three letters may be written in $6 \times 5 \times 4$ ways, and so on.

Therefore the 6 letters can be written down in

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 \text{ or } 720 \text{ ways.}$$

EXERCISE 30

1. How many numbers with two different digits can be formed from the digits 4, 5, 6, 7, 8? Write them all down.

2. Write down all the arrangements of two different letters which can be formed with the letters a, b, c, d.

3. If a cent and a quarter are thrown, each falling either heads or tails, in how many ways can they fall?

4. From a baseball nine a pitcher and a catcher are to be chosen. In how many ways can this be done?

5. A boy has the choice of 5 routes to go to school and 5 to return. In how many different ways may he go and return?

6. In how many ways may two of the letters of the word *winter* be written down?

7. From a class of 20 boys and 10 girls a boy and a girl are to be chosen as leaders. In how many ways may the two be chosen?

8. A, B, C are three towns. From A to B there are 3 routes of travel and from B to C 4. How many ways can I travel from A to C, no two journeys being alike?

9. How many words consisting of a vowel and a consonant can be made from 3 vowels and 8 consonants, if the vowel is always written first?

10. There are 25 single seats in a room. In how many ways can 3 pupils be assigned to seats?

11. In how many ways can I write down three different letters chosen from the letters a, b, c, d, e?

12. In how many ways can 3 prizes be distributed among 20 pupils: (1) if no pupil can receive two prizes; (2) if any pupil may receive all the prizes?

13. From a committee of 10 persons a chairman, a secretary, and a treasurer are to be chosen. In how many ways can this be done?

82. Permutations. Each different arrangement of some or all of a number of things is called a **permutation**.

Thus, the permutations of the letters a, b, c

one at a time are a, b, c;
two at a time are ab, ac, ba, bc, ca, cb;
three at a time are $abc, acb, bac, bca, cab, cba$.

The number of permutations of n things when taken r at a time is represented by the symbol ${}_nP_r$.

Thus, $${}_3P_1 = 3,\ {}_3P_2 = 6,\ {}_3P_3 = 6.$$

83. To find the Number of Permutations of n Different Things taken r at a Time.

Let the n different things be the letters $a, b, c, d, \ldots$

The first letter chosen may be a or b or $c \ldots$, so that there are n ways in which the first letter may be chosen.

$$\therefore \quad {}_nP_1 = n.$$

After the first letter is chosen there are $n - 1$ letters left and any one of these may be chosen as the second letter. There are therefore $n - 1$ choices for the second letter.

Therefore the number of ways of choosing two letters is the product of n and $n - 1$.

$$\therefore \quad {}_nP_2 = n(n - 1).$$

After two letters are chosen there are $n - 2$ letters left. Therefore there are $n - 2$ choices for the third letter.

$$\therefore \quad {}_nP_3 = n(n - 1)(n - 2).$$

Similarly, $${}_nP_4 = n(n - 1)(n - 2)(n - 3),$$

and $${}_nP_5 = n(n - 1)(n - 2)(n - 3)(n - 4).$$

In ${}_nP_5$ there are 5 factors, one corresponding to each thing chosen. Similarly, in ${}_nP_{12}$ there would be 12 factors, the last one of which would be $n - 11$, since the 12th letter

would be chosen from the $n - 11$ letters left after choosing 11 letters.

So ${}_nP_r$ would consist of the product of r factors, the first being n and the last being $n - (r - 1)$ or $n - r + 1$.

$$\therefore \quad {}_nP_r = n(n-1)(n-2) \ldots (n-r+2)(n-r+1).$$

84. To find the Number of Permutations of n Things taken n at a Time, that is, taken all together.

From the preceding

$$\begin{aligned} {}_nP_r &= n(n-1)(n-2) \ldots (n-r+1). \\ \therefore \quad {}_nP_n &= n(n-1)(n-2) \ldots (n-n+1), \\ &= n(n-1)(n-2) \ldots 3 \,.\, 2 \,.\, 1. \end{aligned}$$

That is, ${}_nP_n$ is the product of the n natural numbers from 1 to n. This product is represented by the symbol $\underline{|n}$ or $n\,!$ and is read "factorial n."

$$\therefore \quad {}_nP_n = n(n-1)(n-2) \ldots 3 \,.\, 2 \,.\, 1 = \underline{|n}.$$

Ex. 1.—Find the number of ways of writing down all the letters of the word *second*, (1) where there is no restriction, (2) if the first letter must be *s*, (3) if the first letter must be a vowel, (4) if the second and fourth letters must be vowels.

(1) The result is evidently ${}_6P_6$ or $\underline{|6}$ or 720.

(2) As the first letter must be *s*, there is but one way of choosing the first letter. The other 5 letters may then be written in any order and therefore the result is $1 \times \underline{|5}$ or 120.

(3) There are two choices for the first letter, since there are two vowels. After the vowel is chosen the other 5 letters may be written in any order and therefore the result is $2 \times \underline{|5} = 240$.

(4) There are two ways of writing the second and fourth letters, either *eo* or *oe*. After they are written the other 4 letters may be arranged in $\underline{|4}$ ways.

$\therefore$ the result is $2 \times \underline{|4} = 48$.

Ex. 2.—In how many ways can 10 boys stand in a row if (1) A and B must be together, (2) A and B must not be together?

If there is no restriction the number of ways is evidently $\lfloor\underline{10}$.

(1) If A and B must stand together we might consider them as one person, but we could so consider them in 2 ways as AB or BA. The 9 persons could then be arranged in $\lfloor\underline{9}$ ways.

$$\therefore \quad \text{the result is } 2 \times \lfloor\underline{9} = 725{,}760.$$

Or we might leave A out of consideration, then the other 9 could be arranged in $\lfloor\underline{9}$ ways. In any one of these arrangements A could be placed at either side of B, that is, in 2 ways, and the result as before is $2\lfloor\underline{9}$.

(2) If A is left out the other 9 can be arranged in $\lfloor\underline{9}$ ways. A could then be placed in any one of these arrangements in 8 ways, so that A and B would not be together.

$$\therefore \quad \text{the result is } 8 \times \lfloor\underline{9} = 2{,}903{,}040.$$

Or without restriction the 10 boys can be arranged in $\lfloor\underline{10}$ ways.
But by (1) there are $2\lfloor\underline{9}$ ways in which A and B are together.

$$\therefore \quad \text{the number in which they are not together}$$
$$= \lfloor\underline{10} - 2\lfloor\underline{9} = 10\lfloor\underline{9} - 2\lfloor\underline{9} = 8\lfloor\underline{9}.$$

Ex. 3.—How many numbers of not more than four digits can be formed from the nine digits 1, 2, 3 . . . 9, no digit being repeated in any number?

The number formed with one digit $= {}_9P_1$
The number formed with two digits $= {}_9P_2$
The number formed with three digits $= {}_9P_3$
The number formed with four digits $= {}_9P_4$

$$\therefore \quad \text{the total number} = {}_9P_1 + {}_9P_2 + {}_9P_3 + {}_9P_4$$
$$= 9 + 9 \cdot 8 + 9 \cdot 8 \cdot 7 + 9 \cdot 8 \cdot 7 \cdot 6$$
$$= 3609.$$

85. To find the Number of Permutations of n Things r at a Time when each may be repeated any Number of Times.

Here there are n choices for the first thing, n for the second, n for the third, and so on.

$\therefore$ the required number $= n \cdot n \cdot n \ldots$ to r factors,

$= n^r$.

Ex.—How many numbers of five digits can be formed from the digits 1, 2, 3, 4?

There are 4 choices for the first digit, 4 for the second, and so on.

$\therefore$ the number $= 4^5 = 1024$.

EXERCISE 31

1. Find the numerical values of ${}_{15}P_2$, ${}_7P_4$, ${}_6P_5$.

2. Show that ${}_{10}P_4 = {}_7P_7$; $\lfloor\underline{8} = 56\lfloor\underline{6}$; $\lfloor\underline{n+1} = n(n^2-1)\lfloor\underline{n-2}$; $n\lfloor\underline{n} = \lfloor\underline{n+1} - \lfloor\underline{n}$.

3. In how many ways can 6 books be arranged on a shelf?

4. How many changes can be rung with 7 bells, if all are rung each time?

5. In how many ways can all the letters of the word *Hamilton* be written down?

6. How many words each containing 4 letters can be formed from the letters of the word *number*?

7. If the number of permutations of n things 3 together is one-half of the number of permutations of $3n$ things 2 together, find n.

8. If $\lfloor\underline{n+1} = 56\lfloor\underline{n-1}$, find n.

9. Show that ${}_{n+1}P_{r+1} = (n+1){}_nP_r$; $\dfrac{n^2}{\lfloor\underline{n}} = \dfrac{1}{\lfloor\underline{n-2}} + \dfrac{1}{\lfloor\underline{n-1}}$.

10. If ${}_nP_3 : {}_{n+2}P_5 = 1 : 30$, find n.

11. How many signals can be made with 5 different flags, placed one above another, using (1) 3 at a time, (2) any number at a time?

12. Find the number of arrangements of the letters of the word *algebra* in which the two *a*'s are together.

13. In how many ways can n boys stand in a row? In how many if A and B must not be together?

14. In how many ways can 5 boys and 5 girls stand in a row if no two boys are to be together?

15. In how many ways may 8 persons be seated at a round table, (1) when the seats are distinguished, (2) when relative position only is considered?

16. How many numbers each containing 3 digits can be formed from the nine digits, (1) if no digit is repeated in any number, (2) if the digits may be repeated?

17. If the letters of the word *reason* be written in every possible way, how many of them will not begin with *re*?

18. How many numbers between 3000 and 5000 can be formed with the digits 1, 3, 5, 7, 9, no digit being repeated?

19. Find the total number of permutations (repetitions allowed) of 6 things, not more than 4 being taken at a time.

20. At an examination there are 7 papers, 3 of which are in mathematics. How many possible time-tables may be made for the examination, if the first paper and the last paper must be a mathematical paper?

21. Find the sum of all the numbers which can be formed from the digits 1, 2, 3, 4, 5, using all the digits without repeating any digit.

86. When we are required to find the number of permutations of a number of things, it is understood that the things are all different. If some of the things are alike the number of permutations is diminished.

Ex.—Find the number of permutations of the letters *aaabcd* taken all together.

If the 3 a's were different (like a_1, a_2, a_3), there would then be 6 different letters and the number of permutations would be $\lfloor 6$.

Any permutation as *baadac* where the 3 a's are alike would make $\lfloor 3$ permutations if the a's were different, by merely permutating the 3 a's, the other letters remaining fixed.

Therefore there are $\lfloor 3$ times as many permutations when the a's are different as there are when there are 3 a's alike.

$$\therefore \quad \text{the required number} = \frac{\lfloor 6}{\lfloor 3} = 6 \cdot 5 \cdot 4 = 120.$$

Similarly, the number of permutations of the letters *aaabbcd* taken all together is:

$$\frac{\lfloor 7}{\lfloor 3 \lfloor 2} = 420.$$

87. To find the Number of Permutations of n Letters, taken all together, in which p of the Letters are a's, q are b's, r are c's and the rest are different.

Let x be the required number of permutations. If we take any one of these permutations and suppose the p a's to be changed into p letters different from each other and from the rest of the letters, this one permutation would make $\lfloor\underline{p}$ permutations by changing the order of the p letters without disturbing the other letters.

Therefore, if the a's were different the number of permutations would be $x\lfloor\underline{p}$.

Similarly, if the q b's were changed to different letters, the number of permutations would be multiplied by $\lfloor\underline{q}$, and if the r c's were different it would be multiplied again by $\lfloor\underline{r}$.

But since the letters would now be all different the number of permutations would be $\lfloor\underline{n}$.

$$\therefore \quad x\lfloor\underline{p}\ \lfloor\underline{q}\ \lfloor\underline{r} = \lfloor\underline{n}$$

$$\therefore \qquad x = \frac{\lfloor\underline{n}}{\lfloor\underline{p}\ \lfloor\underline{q}\ \lfloor\underline{r}}.$$

Ex. 1.—Find the number of permutations of the letters of the word *Mississippi* taken all together.

In this word there are 11 letters of which 4 are i's, 4 are s's and 2 are p's.

$\therefore$ by the formula the number of permutations

$$= \frac{\lfloor\underline{11}}{\lfloor\underline{4}\ \lfloor\underline{4}\ \lfloor\underline{2}} = 34650.$$

Ex. 2.—How many numbers can be formed by using all of the digits 1, 2, 3, 4, 5, 6 without changing the order of the odd digits?

Since the odd digits are not to be changed in order, the number is the same as if the three odd digits were alike and is therefore $\frac{\lfloor\underline{6}}{\lfloor\underline{3}} = 120$.

EXERCISE 32

1. Find the number of permutations of the letters of the words (*a*) *perimeter*, (*b*) *Malacca*, (*c*) *indivisible*, taken all together.

2. How many different numbers can be formed using all of the digits 1, 1, 1, 2, 2, 3, 4 ?

3. In how many ways can the letters of the word *Canada* be written, if each begins with *C* and ends with *a* ?

4. How many more permutations are there of the letters of the word *alter* than of the word *altar*, all the letters being used in each case ?

5. In how many ways can 3 similar copies of an algebra, 4 of a geometry and 2 of an arithmetic be given to 9 boys, if each boy gets one ?

6. How many even numbers each with 6 digits can be formed with the digits 2, 3, 3, 5, 5, 5 ? How many odd numbers ?

7. How many permutations are there of the letters of the word *serene* taken all together ? How many of them begin with *e* ? With 2 *e*'s ? With 3 *e*'s ? With just 2 *e*'s ?

8. There are 8 letters of which some are *a*'s and the rest are all different. The number of different arrangements of the letters taking them all together is 1680. How many are *a*'s ?

9. In how many ways can 7 quarters and 5 ten-cent pieces be given to 12 boys, one coin to each ?

10. How many arrangements of 11 letters can be formed from the letters in $a^3b^5c^2d$?

11. How many numbers greater than a million can be formed by using the digits 2, 5, 0, 3, 5, 2, 5 ?

12. Find the number of ways in which the 9 digits may be written down, if the odd digits are in ascending order and also the even digits.

13. Without changing the order of the vowels, how many arrangements can be made of the letters of the word *advise* ? Of the word *advised* ?

14. In going to school a boy walks 5 blocks east and 4 blocks south. How many different ways can he go, if he never walks more than 9 blocks ?

15. How many numbers can be formed by using all of the digits 1, 2, 3, 3, 3, 4, 4, 5, 6, so that the odd digits always occupy the odd places?

88. Combinations. Each different group or selection which can be made from a number of things, in which the order of arrangement in the group is not considered, is called a **combination.**

Thus, the combinations of the letters a, b, c, d taken
one at a time are a, b, c, d;
two at a time are ab, ac, ad, bc, bd, cd;
three at a time are abc, abd, acd, bcd;
four at a time are $abcd$.

The number of combinations of n things taken r at a time is represented by the symbol ${}_nC_r$.

Thus, $${}_4C_1 = 4,\ {}_4C_2 = 6,\ {}_4C_3 = 4,\ {}_4C_4 = 1.$$

It is important to be able to distinguish when the number of permutations is required and when the number of combinations.

Thus, if we wish to know in how many ways a committee of two can be selected from five persons, it is clear that it is a problem in combinations as it is immaterial in what order the two persons are selected.

But if we wish to know in how many ways a chairman and a secretary can be selected from a committee of five, it is evidently a problem in permutations as it is a different selection if A is chosen as chairman and B as secretary than if B is chosen chairman and A secretary.

Ex.—Find the number of combinations of the letters $abcde$ taken 3 at a time.

One of the combinations would be abc. This one combination would make $\underline{|3}$ permutations by changing the order of the letters a, b, c.

Similarly, the combination *ade* would make $\lfloor 3$ permutations, and so on.

That is, for every combination of 3 letters there would be $\lfloor 3$ permutations, and therefore the number of permutations is $\lfloor 3$ times the number of combinations.

But $\quad {}_5P_3 = 5 \cdot 4 \cdot 3, \qquad \therefore \quad {}_5C_3 = 5 \cdot 4 \cdot 3 \div \lfloor 3 = 10.$

The 10 combinations are :

abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde.

89. To find the Number of Combinations of n Different Things taken r at a Time.

Each combination of r things would make $\lfloor r$ permutations by changing the order of the things in the combination, and therefore there are $\lfloor r$ times as many permutations as combinations,

or $$ {}_nC_r \times \lfloor r = {}_nP_r.$$

But $$ {}_nP_r = n(n-1)(n-2) \ldots (n-r+1),$$

$$\therefore \quad {}_nC_r = \frac{n(n-1)(n-2) \ldots (n-r+1)}{\lfloor r}.$$

Thus, the number of combinations of 20 things 3 at a time is

$${}_{20}C_3 = \frac{20 \cdot 19 \cdot 18}{\lfloor 3} = 1140.$$

Similarly, $${}_{20}C_{14} = \frac{20 \cdot 19 \cdot 18 \ldots 8, 7}{\lfloor 14}$$

The last result may be put into a simpler form by multiplying numerator and denominator by $\lfloor 6$, when the numerator becomes $\lfloor 20$.

$$\therefore \quad {}_{20}C_{14} = \frac{\lfloor 20}{\lfloor 14 \, \lfloor 6}.$$

Similarly, in the general formula

$$ {}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} $$

$$ = \frac{n(n-1)\ldots(n-r+1)}{\lfloor r} \times \frac{(n-r)(n-r-1)\ldots 3.2.1}{\lfloor n-r} . $$

$$ \therefore \quad {}_nC_r = \frac{\lfloor n}{\lfloor r \ \lfloor n-r} . $$

If we wish to find the numerical value of a quantity like $\frac{\lfloor 20}{\lfloor 14 \ \lfloor 6}$, the larger factorial in the denominator should be divided into the numerator.

Thus, $$ \frac{\lfloor 20}{\lfloor 14 \ \lfloor 6} = \frac{20 . 19 . 18 . 17 . 16 . 15}{\lfloor 6} = 38760. $$

Ex.—Find the number of (1) lines, (2) triangles, which can be formed by joining 15 points in every possible way, if no three of the points are in the same straight line.

(1) A line will be formed by joining any two points which may be chosen, and every two points will determine a different line from every other two and therefore the number of lines is the same as the number of ways in which 2 things can be chosen from 15.

The line which joins the points A and B is evidently the same as the line which joins B and A, and therefore the problem is one in combinations.

Therefore the number of lines $= {}_{15}C_2 = \frac{15 . 14}{2} = 105.$

(2) To form a triangle the three angular points must be selected and joined.

Therefore the number of triangles $= {}_{15}C_3 = \frac{15 . 14 . 13}{\lfloor 3} = 455.$

90. The Number of Combinations of n Different Things taken r at a Time is the same as the Number $n - r$ at a Time.

For $$ {}_nC_r = \frac{\lfloor n}{\lfloor r \,\lfloor n - r}. $$

$$ \therefore \quad {}_nC_{n-r} = \frac{\lfloor n}{\lfloor n - r \,\lfloor n - (n - r)} = \frac{\lfloor n}{\lfloor n - r \,\lfloor r}. $$

$$ \therefore \quad {}_nC_r = {}_nC_{n-r}. $$

Or, without using the formula the truth of the theorem will at once be seen by observing that when r things are chosen there are $n - r$ things left. That is, to every combination of r things there corresponds a combination of $n - r$ things.

Thus, if I wish to select 18 men from 20, the simplest way would be to decide first on the 2 who were not to be selected.

That is, $$ {}_{20}C_{18} = {}_{20}C_2 $$

The converse is also true, that if ${}_nC_a = {}_nC_b$, then $a + b = n$.

Since $$ {}_nC_r = {}_nC_{n-r}, \text{ then } {}_nC_n = {}_nC_0. $$

But ${}_nC_n = 1$, so that when the symbol ${}_nC_0$ is used its value will be considered as 1.

Ex.—In how many ways can a team of 9 players be selected from 13 players?

The number of ways $= {}_{13}C_9 = {}_{13}C_4 = \dfrac{13 \cdot 12 \cdot 11 \cdot 10}{\lfloor 4} = 715.$

If the factorial form is used, it is immaterial whether we use ${}_{13}C_9$ or ${}_{13}C_4$, for each is $\dfrac{\lfloor 13}{\lfloor 9 \,\lfloor 4}$, and in simplifying the $\lfloor 9$ should be divided out.

EXERCISE 33

1. Find the number of combinations of (*a*) 5 things 3 at a time; (*b*) 8 things 6 at a time; (*c*) 20 things 4 at a time.

2. Find the numerical values of ${}_6C_3$, ${}_{15}C_{12}$, ${}_{13}C_5$, ${}_{10}C_7$.

3. Find the value of ${}_6C_r$ when $r = 1, 2, 3, 4, 5$ or 6. Which of these is greatest? For what value of r is ${}_7C_r$ greatest?

4. How many different committees can be formed by selecting 3 persons from 12?

5. If the number of combinations of $2n$ things 3 at a time is 11 times the number of combinations of n things 3 at a time, find n.

6. If ${}_{n+1}P_3 : {}_nC_4 = 36 : 5$, find n.

7. If ${}_nC_3 = {}_nC_4$, find n.

8. If ${}_nC_5 = {}_nC_{12}$, find ${}_nC_{15}$ and ${}_{20}C_n$.

9. Find the value of ${}_{100}C_{98} - {}_{99}C_{97}$.

10. If 8 points, no three of which are in a straight line, be joined in all possible ways, how many lines are formed?

11. If a polygon has 8 sides, how many diagonals has it?

12. How many diagonals are there in a polygon of n sides?

13. If a polygon has 44 diagonals, how many sides has it?

14. There are n indefinite straight lines no two of which are parallel and no three pass through the same point. How many intersections are there?

15. How many triangles can be formed whose angular points are on n given points, no three of which are in a straight line?

16. There are 20 points all in one straight line and 15 others in a parallel line. How many triangles, having these points as angular points, can be formed by joining the 35 points in all possible ways?

17. In how many ways can 10 things be divided equally between two persons?

18. Out of 9 different books, A is to take 4 and B is to take 3. How many different selections are possible?

19. In how many ways can a party of 16 people be conveyed in 2 vehicles, one of which will not hold more than 8 people and the other not more than 10?

91. Ex. 1.—In how many ways can a committee of 5 be selected from 20 persons? In how many if a particular

person A must be selected? In how many if A must not be selected?

$$\text{The total number} = {}_{20}C_5 = \frac{\lfloor 20}{\lfloor 5 \,\lfloor 15} = 15504.$$

If A must be chosen, then we are really selecting 4 persons from the other 19. Therefore the number, if A must be selected,

$$= {}_{19}C_4 = \frac{\lfloor 19}{\lfloor 4 \,\lfloor 15} = 3876.$$

If A must not be chosen, then we are really selecting 5 from the other 19. Therefore the number in which A is not selected

$$= {}_{19}C_5 = \frac{\lfloor 19}{\lfloor 5 \,\lfloor 14} = 11628.$$

If we add together the number in which A is chosen and the number in which he is not chosen, we evidently have the whole number of combinations without restriction.

That is, $${}_{20}C_5 = {}_{19}C_4 + {}_{19}C_5,$$

and in the same way we may show that

$${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r;$$

or thus,

$${}_{n-1}C_{r-1} + {}_{n-1}C_r = \frac{\lfloor n-1}{\lfloor r-1 \,\lfloor n-r} + \frac{\lfloor n-1}{\lfloor r \,\lfloor n-r-1}$$

$$= \frac{r\lfloor n-1}{\lfloor r \,\lfloor n-r} + \frac{(n-r)\lfloor n-1}{\lfloor r \,\lfloor n-r} = \frac{n\lfloor n-1}{\lfloor r \,\lfloor n-r} = \frac{\lfloor n}{\lfloor r \,\lfloor n-r} = {}_nC_r.$$

Ex. 2.—In how many ways can a party of 6 privates and 2 officers be chosen from 8 privates and 4 officers?

The 6 privates can be selected from 8 in ${}_8C_6$ ways.
The 2 officers can be selected from 4 in ${}_4C_2$ ways.
Therefore the number of ways the party can be made up

$$= {}_8C_6 \times {}_4C_2 = 28 \times 6 = 168.$$

Ex. 3.—How many words consisting of 3 consonants and 2 vowels can be formed from 21 consonants and 5 vowels?

The 3 consonants can be selected in ${}_{21}C_3$ ways and the 2 vowels in ${}_5C_2$ ways.

Therefore 3 consonants and 2 vowels can be selected in ${}_{21}C_3 \times {}_5C_2$ ways.

But the 5 letters in any group can be arranged in $\lfloor 5$ ways, or the same 5 letters will make $\lfloor 5$ words.

Therefore the number of words $= {}_{21}C_3 \times {}_5C_2 \times \lfloor 5 = 1{,}596{,}000.$

Ex. 4.—In how many ways can a committee of 5 be chosen from 4 masters and 10 pupils, so as always to include at least one master?

The committee can be formed so as to contain (1) 4 masters and 1 pupil, (2) 3 masters and 2 pupils, (3) 2 masters and 3 pupils, (4) 1 master and 4 pupils.

Therefore the total number of ways of forming the committee

$$\begin{aligned} &= {}_4C_4 \times {}_{10}C_1 + {}_4C_3 \times {}_{10}C_2 + {}_4C_2 \times {}_{10}C_3 + {}_4C_1 \times {}_{10}C_4 \\ &= 1 \times 10 + 4 \times 45 + 6 \times 120 + 4 \times 210 \\ &= 10 + 180 + 720 + 840 = 1750. \end{aligned}$$

92. Ex. 1.—Find the total number of combinations of 5 different things chosen any number at a time.

The number would evidently be

$$\begin{aligned} &{}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 \\ &= 5 + 10 + 10 + 5 + 1 = 31. \end{aligned}$$

Or we might reason thus:

Each thing may be selected or not, that is, there are 2 ways of disposing of each thing.

Therefore the total number is $2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$

This includes the case in which all are rejected, and if this is not counted the result is 31.

Thus $${}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 2^5 - 1.$$

In the same way we may show that

$$_nC_1 + {}_nC_2 + {}_nC_3 + \ldots + {}_nC_n = 2^n - 1.$$

Ex. 2.—A purse contains coins of the following values: 1 cent, 5 cents, 10 cents, 25 cents, 50 cents and 1 dollar, there

being only one of each. How many sums of different values can be made from them?

Since there are coins of 6 different values, the number of sums is $2^6 - 1$ or 63. It is evident that no combination of these coins would make the same sum as any other combination.

Would this be true if there was also a 20-cent piece?

93. To find the Total Number of Ways in which a Selection can be made from $p + q + r + \ldots$ Things, in which p are alike of one kind, q of another, r of another, and so on.

The p things may be disposed of in $p + 1$ ways, for we may choose 0, 1, 2, . . . or p of them. Similarly, the q-like things may be disposed of in $q + 1$ ways, and so on.

Therefore they may all be disposed of in

$$(p + 1)(q + 1)(r + 1) \ldots \text{ ways.}$$

This includes the case in which all are left out, therefore the number of ways in which a selection can be made is

$$(p + 1)(q + 1)(r + 1) \ldots - 1.$$

Ex.—How many factors are there of 360?

Since $360 = 2^3 . 3^2 . 5$, a factor may be obtained by multiplying together any number of factors selected from 2, 2, 2, 3, 3, 5.

The number of selection is $4 \times 3 \times 2 = 24$.

One of these would be the case in which all the factors are chosen which would give the factor $2^3 . 3^2 . 5$ or 360.

Another in which none of the factors is chosen would correspond to $2^0 . 3^0 . 5^0$ or 1.

If 1 and 360 are not counted the number of factors is 22. Verify this by writing down the 22 factors of 360.

EXERCISE 34

1. From 8 men and 9 women a committee is to be formed comprising 5 of each sex. In how many ways can it be done?

2. Out of 16 consonants and 5 vowels, how many words can be formed containing 4 consonants and 1 vowel? In how many would the vowel be the middle letter?

3. In how many ways can a party of 4 men be selected from 12 men? In how many if (1) a particular man is always included, (2) if a particular man is excluded?

4. A box contains 15 balls numbered from 1 to 15. In how many ways can 6 balls be selected if 1 is included and 2 and 3 are excluded?

5. Find the number of ways in which a picket of 6 men can be chosen from 15 men, (1) if A and B must be chosen, (2) if A and B must not be chosen, (3) if A must be chosen but B must not.

6. A person has 20 friends, 10 of whom are relatives. In how many ways may he invite 15 guests from among them so that only 8 of them are relatives?

7. In how many ways can a watch of 5 be chosen from 3 officers and 12 men so as always to include at least one officer?

8. A committee of 7 is to be chosen from 9 Conservatives and 8 Liberals, each party being represented on the committee. In how many ways may it be done so that there will be a Conservative majority on the committee?

9. Prove that

$$_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$$

and that $$_{n+1}C_r = {}_{n-1}C_r + 2 \times {}_{n-1}C_{r-1} + {}_{n-1}C_{r-2}.$$

10. In a bag are 8 white and 6 red balls, all different. In how many ways may 6 balls be drawn out, so that in each case there may be at least 2 red balls?

11. In how many ways can a selection be made from 10 different books? In how many if at least 2 books must be chosen?

12. Show that in $_{3n}C_n$ the number of combinations in which a particular thing occurs is $\frac{1}{3}$ of the whole number of combinations.

13. How many different weights can be formed from the following weights: 1 lb., 3 lb., 5 lb., 10 lb.?

14. In how many ways can I select one volume or more from 5 similar copies of an algebra and 6 similar copies of an arithmetic?

15. I have 6 different text-books and 3 copies of each. In how many ways can a selection be made from them?

16. Find the number of integral factors of 1400. Of 1260.

94. Find the value of r for which the number of combinations of (1) 8 things, (2) 9 things is greatest.

The number of combinations of 8 things 1, 2, 3 . . . 8 at a time are

$$ {}_8C_1,\ {}_8C_2,\ {}_8C_3,\ {}_8C_4,\ {}_8C_5,\ {}_8C_6,\ {}_8C_7,\ {}_8C_8, $$

or $$ 8,\ 28,\ 56,\ 70,\ 56,\ 28,\ 8,\ 1, $$

and of 9 things 1, 2, 3, . . . 9 at a time are

$$ {}_9C_1,\ {}_9C_2,\ {}_9C_3,\ {}_9C_4,\ {}_9C_5,\ {}_9C_6,\ {}_9C_7,\ {}_9C_8,\ {}_9C_9, $$

or $$ 9,\ 36,\ 84,\ 126,\ 126,\ 84,\ 36,\ 9,\ 1. $$

Therefore ${}_8C_r$ is greatest when $r = 4$ or $\frac{8}{2}$, and ${}_9C_r$ is greatest when $r = 4$ or 5, or $\frac{9-1}{2}$ or $\frac{9+1}{2}$.

95. To find the Value of r for which ${}_nC_r$ is greatest.

$$ {}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{\lfloor r}. $$

$$ {}_nC_{r-1} = \frac{n(n-1)(n-2)\ldots(n-r+2)}{\lfloor r-1} $$

$$ \therefore\quad {}_nC_r = {}_nC_{r-1} \times \frac{n-r+1}{r} = {}_nC_{r-1} \times \left(\frac{n+1}{r} - 1\right). $$

Since $\frac{n+1}{r} - 1$ decreases as r increases, the value of ${}_nC_r$ will continually increase until $\frac{n+1}{r} - 1$ becomes 1 or less than 1, after which the value of ${}_nC_r$ will continually decrease.

Thus, in finding ${}_{10}C_1,\ {}_{10}C_2,\ {}_{10}C_3$ . . . the successive multipliers by which each is formed from the preceding one are

$$ \frac{9}{2},\ \frac{8}{3},\ \frac{7}{4},\ \frac{6}{5},\ \frac{5}{6},\ \frac{4}{7},\ \frac{3}{8},\ \frac{2}{9},\ \frac{1}{10}. $$

Although the multipliers continually decrease the values of ${}_{10}C_r$ will increase until the multiplier become less than 1, which happens in finding ${}_{10}C_6$ from ${}_{10}C_5$.

Since ${}_{10}C_6 = {}_{10}C_5 \times \frac{5}{6}$, it follows that ${}_{10}C_r$ is greatest when $r = 5$.

Similarly, ${}_nC_r > {}_nC_{r-1}$ as long as $\frac{n+1}{r} - 1 > 1$,

or $\frac{n+1}{r} > 2$,

or $\frac{n+1}{2} > r$,

or $r < \frac{n+1}{2}$.

(1) If n is even, the greatest value that r can have and be less than $\frac{n+1}{2}$ is $\frac{n}{2}$.

$\therefore$ **${}_nC_r$ is greatest when $r = \frac{n}{2}$, if n is even.**

(2) If n is odd, then $\frac{n+1}{2}$ is an integer, and when $r = \frac{n+1}{2}$, ${}_nC_r$ will be equal to ${}_nC_{r-1}$ and these two will be greater than any other.

$\therefore$ **${}_nC_r$ is greatest when $r = \frac{n+1}{2}$ or $\frac{n-1}{2}$, when n is odd.**

96. Ex. 1.—In how many ways can 12 different books be divided among A, B and C, giving 3 to A, 4 to B, 5 to C?

A may receive 3 books selected from 12 in ${}_{12}C_3$ ways, B may receive 4 from the remaining 9 in ${}_9C_4$ ways, and C the remaining 5 in 1 way.

$\therefore$ the total number $= {}_{12}C_3 \times {}_9C_4 = \frac{\lfloor 12}{\lfloor 3 \lfloor 9} \times \frac{\lfloor 9}{\lfloor 4 \lfloor 5} = \frac{\lfloor 12}{\lfloor 3 \lfloor 4 \lfloor 5}$.

Ex. 2.—In how many ways can 12 books be divided into 3 equal parcels?

Reasoning as before the number would be $\frac{\lfloor 12}{\lfloor 4 \lfloor 4 \lfloor 4}$.

Since the parcels are not distinguished as they would be if given to different persons as in Ex. 1, and since the 3 parcels can be arranged in $\lfloor 3$ ways, the correct result in this problem is $\frac{\lfloor 12}{\lfloor 4 \lfloor 4 \lfloor 4 \lfloor 3}$.

Verify this by showing that 4 books can be divided equally into 2 parcels in only 3 ways but can be divided equally between 2 persons in 6 ways.

97. Ex.—Find the number of (1) combinations, (2) permutations of the letters of the word *Toronto* taken 5 at a time.

In this word there are 3 *o*'s, 2 *t*'s, *r*, *n*.

The groups of 5 letters might be made up in four different ways :

(1) 3 letters alike and 2 alike. (2) 3 alike and 2 different. (3) 2 alike, 2 alike, 1 different. (4) 2 alike and 3 different.

(1) may be chosen in 1×1 or 1 way; (2) in 1×3 or 3 ways; (3) in $1 \times 1 \times 2$ or 2 ways, and (4) in 2×1 or 2 ways.

$\therefore$ the number of combinations $= 1 + 3 + 2 + 2 = 8$.

In (1) there are 5 letters of which 3 are alike, and 2 alike, therefore each combination in (1) will make $\frac{\lfloor 5}{\lfloor 3 \lfloor 2}$ or 10 permutations.

$\therefore$ from (1) the number of permutations is 10, from (2) 3×20 or 60, from (3) 2×30 or 60, and from (4) 2×60 or 120.

$\therefore$ the number of permutations $= 250$.

EXERCISE 35

1. What should be the value of r to give the greatest number of combinations in each of the following cases :

$${}_{20}C_r,\ {}_{17}C_r,\ {}_{2n}C_r,\ {}_{2n+1}C_r\ ?$$

2. There are 20 different articles offered for sale at 5 cents each. How much money should I spend so as to have the greatest number of choices ?

3. A committee of 8 is to be chosen from 12 men and 7 women. Show that there is the greatest number of choices when 5 men and 3 women are chosen.

4. In how many ways can 20 different books be divided equally between A and B ? In how many ways can they be divided into two equal parcels ?

5. In how many ways can $m + n$ things be divided into two unequal groups containing m and n things?

6. In how many ways can a pack of 52 cards be divided equally (1) among 4 players, (2) into 4 heaps?

7. Find the number of combinations and permutations of the letters of the word *Canada* taken 4 at a time. Also of the word *amalgamate* taken 5 at a time.

8. There are $n - 1$ sets containing $2a, 3a, \ldots, na$ different things respectively. Find the number of combinations that can be formed from these sets by taking a from the first, $2a$ from the second, . . . , $(n - 1)a$ from the last.

EXERCISE 36 (Miscellaneous Examples)

1. How many words of three letters, in which the first and third letters are different consonants and the second letter a vowel, can be formed from 20 consonants and 5 vowels?

2. How many different connections are necessary in a telephone exchange having 200 subscribers?

3. In a town there are three different letter boxes. In how many ways can 5 letters be posted?

4. A captain has 8 different signal flags. How many signals has he at his disposal, each consisting of 3 flags one above another?

5. In a group of 12 hockey players only one can keep goal. In how many ways may a team of 7 be chosen from the 12 so as to include the goal-keeper in every choice?

6. There are 15 different kinds of cakes at a confectioner's. I order two particular kinds and any other three which the confectioner chooses. How many different assortments may I receive?

7. On the circumference of a circle 11 points are taken and joined in all possible ways. Find the number of chords thus formed, and the number of triangles having chords for sides.

8. A man has 15 different books which he wishes to give to 5 friends, giving 3 to each. In how many ways can this be done?

9. In how many ways can 5 different books be divided between 2 boys, if each boy receives at least one?

10. In how many different ways may the letters of the word *flowering* be arranged (1) without changing the position of any vowel, (2) without changing the order of the vowels?

11. In how many numbers of 4 digits, in the decimal system, does the digit 6 occur?

12. 22 boys wish to divide themselves into two equal groups for a game of football. If two of the boys are brothers, in how many ways can the two teams be made up so that the brothers will not play on the same side?

13. Find the total number of permutations, in which repetitions are allowed, of n things not more than r being taken at a time.

14. There are n different points in the same plane, no three of which are in the same straight line except p of them which are all in the same straight line. How many (1) lines, (2) triangles can be formed by joining them in all possible ways?

15. Find how many numbers there are between 10,000 and 100,000 of which the first and last digits are odd and no one contains any repeated digit.

16. If $_{2n}C_{n-1} : {}_{2n-2}C_n = 132 : 35$, find n.

17. Show that $_{2n}C_n = \dfrac{1 \cdot 3 \cdot 5 \ldots 2n-1}{\lfloor n} \cdot 2^n$.

18. In a signaling system using dots and dashes, how many signals can be made using not more than 3 dots and not more than 2 dashes?

19. How many combinations can be made of the letters of the word *concurrent* taken 4 at a time?

20. How many (1) selections, (2) arrangements can be made of seven out of the eight letters of the word *football*?

21. Prove that the number of ways in which p positive signs and r negative signs can be placed in a row so that no two negative signs are together is $_{p+1}C_r$.

22. A crew of 8 men is to be chosen to row an eight-oared boat. If there are 11 men, 5 of whom can row on the stroke side only, 4 on the bow side only, and the remaining 2 on either side, in how many ways may the boat be manned?

23. Prove that $_{n+2}C_{r+1} = {}_nC_{r+1} + 2_nC_r + {}_nC_{r-1}$.

24. If 12 different things are to be put up in 3 packages, one of 3, one of 4, one of 5 things, in how many ways can it be done?

25. If 10 men and 7 women apply for 8 different positions, 3 of which must be filled by men and 2 by women, the remaining 3 being filled by either men or women, in how many ways may the positions be filled?

26. In how many ways may different sets of tennis be made up from a party of 5 girls and 7 boys, each set consisting of 2 girls and 2 boys, the girls not playing on the same side?

27. Show that the greatest number of combinations of $2n$ things is double the greatest number of combinations of $2n - 1$ things.

28. If ${}_{n+1}C_{r+2} = 9 \times {}_nC_{r+1} = 90 \times {}_{n-1}C_r$, find n and r.

29. Find the number of combinations, taken $2n$ at a time, of $3n$ things of which $2n$ are alike and the others different.

30. Show that ${}_nC_0{}^2 + {}_nC_1{}^2 + {}_nC_2{}^2 + \ldots + {}_nC_n{}^2 = \frac{\lfloor 2n}{\lfloor n \lfloor n}$.

(Divide $2n$ things into two equal groups. Select n from the first group and 0 from the second, $n - 1$ from the first and 1 from the second, and so on.)

CHAPTER XII

THE BINOMIAL THEOREM. POSITIVE INTEGRAL INDICES

98. Law of Multiplication. The product of

$$a + b \text{ and } c + d \text{ is } ac + bc + ad + bd.$$

If this product be multiplied by $x + y$ the result is

$$acx + bcx + adx + bdx + acy + bcy + ady + bdy.$$

In this last product it will be seen that each partial product is of three dimensions, that is, contains three letters, one of which is taken from each of the three factors which were multiplied together.

Thus the term acx is obtained by choosing the first term from each factor, bcy by choosing b from $a + b$, c from $c + d$ and y from $x + y$.

The eight terms in the product contain *all* the possible ways of obtaining a partial product by multiplying together three letters, one of which is selected from each of the given factors.

In no case is a term of the product obtained by choosing two letters from the same factor. Thus both a and b do not occur in any term in the product, nor do c and d, nor x and y.

That is, the terms in the product of $a + b$, $c + d$, $x + y$ are obtained by finding the product in every possible way of three letters, *one and only one* being selected from each of the given factors.

When any number of factors are multiplied together, the same law holds with regard to the product.

Thus, in the product of the ten factors

$$(a_1 + b_1)(a_2 + b_2)(a_3 + b_3) \ldots (a_{10} + b_{10}),$$

every term will contain ten letters, one and only one being selected from each of the ten given factors.

Ex. 1.—Find the product of $x + a$, $x + b$, $x + c$.

Each term in the product will be of three dimensions.

If we select x from each factor we obtain the term x^3.

If we select x from the first and second and c from the third we get x^2c, while if we select x from the first and third and b from the second we get x^2b, and selecting x from the second and third and a from the first we get x^2a.

There are therefore three terms containing x^2, being

$$x^2a + x^2b + x^2c \text{ or } x^2(a + b + c).$$

If we select x from the first factor, b from the second and c from the third we get the term xbc. Similarly, we get xac and xab.

There are therefore three terms containing x, being

$$xab + xac + xbc \text{ or } x(ab + ac + bc).$$

If no x is selected, but a, b and c are chosen, we get the term abc.

$$\therefore \quad (x+a)(x+b)(x+c) = x^3 + x^2(a+b+c) + x(ab+ac+bc) + abc.$$

Verify this result by ordinary multiplication.

Ex. 2.—Expand $(a + x)^4$.

Here we are required to find the product of four factors, each being $a + x$.

Each term in the product will be of four dimensions.

The literal parts of the terms must be

$$a^4, a^3x, a^2x^2, ax^3, x^4.$$

The 1st term in the expansion is formed by selecting a from each factor, and this can be done in only one way.

The coefficient of a^4 is therefore unity.

The term containing a^3x is formed by selecting a from any three

of the factors and x from the remaining factor. Since 3 a's can be selected from 4 a's in 4 ways, the coefficient of a^3x is 4.

$\therefore$ the 2nd term in the expansion is $4\,a^3x$.

The term containing a^2x^2 is obtained by selecting a from two of the factors and x from each of the remaining factors. But 2 a's can be selected from 4 a's in ${}_4C_2$ or 6 ways

$\therefore$ the 3rd term is $6a^2x^2$.

The term containing ax^3 is obtained by selecting a from one factor and x from each of the remaining factors, and this can be done in 4 ways.

$\therefore$ the 4th term is $4ax^3$.

The term containing x^4 can be formed in only one way by selecting x from each of the four factors.

$$\therefore \quad (a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

Verify by multiplication.

Ex. 3.—Find the first three terms in $(a + b)^{20}$.

The 1st term is formed by selecting a from each of the 20 factors, and this can be done in only one way,

$\therefore$ the 1st term is a^{20}.

The 2nd term is formed by selecting a from 19 of the factors and b from the remaining factor. This can be done in ${}_{20}C_{19}$ ways. But ${}_{20}C_{19} = {}_{20}C_1 = 20$,

$\therefore$ the 2nd term is $20a^{19}b$.

The 3rd term is formed by selecting a from 18 factors and b from the others. This can be done in ${}_{20}C_{18}$ ways.

But $${}_{20}C_{18} = {}_{20}C_2 = \frac{20 \cdot 19}{1 \cdot 2} = 190,$$

$\therefore$ the 3rd term is $190a^{18}b^2$.

$$\therefore \quad (a + b)^{20} = a^{20} + 20a^{19}b + 190a^{18}b^2 + \ldots$$

Ex. 4.—Find the term containing x^5y^4 in $(x + y)^9$.

This term is formed by selecting x from 5 of the factors and y from the remaining factors. This can be done in ${}_9C_5$ ways.

$\therefore$ the required term $= {}_9C_5x^5y^4 = 126x^5y^4$.

EXERCISE 37

Write down the product of :

1. $(x + a)(x + b)$.
2. $(x + 1)(x + 2)(x + 3)$.
3. $(x + a)(x + b)(x + c)(x + d)$.
4. $(a + b)^3$.
5. Expand $(x + y)^4$ and $(x + a)^5$.
6. Find the first three terms in $(a + x)^{10}$ and in $(x + y)^{15}$.
7. Find the 3rd term in $(a + b)^{25}$ and in $(m + n)^{15}$.
8. In the expansion of $(a + x)^7$, the term which contains a^4 will contain what power of x? Which term is this, and what is its coefficient?
9. Find the term containing x^5a^7 in $(x + a)^{12}$.
10. How many terms are there in the expansions of

$$(m + n)^4,\ (a + b)^7,\ (x + y)^{20}?$$

99. The Binomial Theorem. We know by multiplication that

$$(a + x)^2 = a^2 + 2ax + x^2.$$
$$(a + x)^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$
$$(a + x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

We now wish to find the expansion of $(a + x)^n$, where n is any positive integer.

$(a + x)^n$ is the product of n factors each equal to $a + x$.

Each term in the product will be of n dimensions, one and only one factor being selected from each of the quantities multiplied.

The terms in the expansion, omitting the coefficients, will be

$$a^n,\ a^{n-1}x,\ a^{n-2}x^2,\ \ldots,\ ax^{n-1},\ x^n.$$

The 1st term is obtained by selecting a from each of the given factors, and this can be done in only one way.

$$\therefore \text{ the 1st term } = a^n.$$

The 2nd term is obtained by selecting a from $n - 1$

factors and x from the remaining factor. The $n-1$ a's can be selected from n a's, or one x from n x's, in ${}_nC_1$ or n ways.

$$\therefore \quad \text{the 2nd term} = {}_nC_1 a^{n-1}x = na^{n-1}x.$$

The 3rd term is found by selecting a from $n-2$ factors and x from each of the remaining factors, and this can be done in ${}_nC_{n-2}$ or ${}_nC_2$ or $\dfrac{n(n-1)}{2}$ ways.

$$\therefore \quad \text{the 3rd term} = {}_nC_2 a^{n-2}x^2 = \frac{n(n-1)}{2}a^{n-2}x^2.$$

Similarly, the 4th term

$$= {}_nC_3 a^{n-3}x^3 = \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3.$$

The term containing $a^{n-r}x^r$ will be found by selecting a from $n-r$ factors and x from each of the remaining factors, and this can be done in ${}_nC_{n-r}$ or ${}_nC_r$ ways.

$\therefore$ the term containing $a^{n-r}x^r$

$$= {}_nC_r a^{n-r}x^r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}a^{n-r}x^r.$$

The last term is formed by selecting x from each of the given factors, and this can be done in ${}_nC_n$ or 1 way.

$$\therefore \quad \text{the last term} = {}_nC_n x^n = x^n.$$

$$\therefore \quad (a+x)^n = a^n + {}_nC_1 a^{n-1}x + {}_nC_2 a^{n-2}x^2 + {}_nC_3 a^{n-3}x^3 + \ldots$$

$$+ {}_nC_r a^{n-r}x^r + \ldots + {}_nC_{n-1}ax^{n-1} + {}_nC_n x^n.$$

$$\text{or} \quad (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 + \ldots$$

$$+ \frac{n(n-1)(n-2)\ldots(n-r+1)}{\underline{|r}}a^{n-r}x^r + \ldots + nax^{n-1} + x^n.$$

This statement is called the **Binomial Theorem**. It should be noted that, as yet, the theorem has been established only for positive integral values of n.

100. The following deductions from the theorem are worthy of attention :

(1) Since the indices of x in the successive terms are 0, 1, 2, . . . n, it follows that the number of terms in the expansion of $(a + x)^n$ is $n + 1$.

(2) The 3rd term contains x^2, the 4th x^3, the 10th x^9, the $(r + 1)$th x^r. That is, the index of x in any term is one less than the number of the term.

(3) The coefficient of the term containing x^2 is ${}_nC_2$, of the term containing x^3 is ${}_nC_3$, of x^4 is ${}_nC_4$, of x^r is ${}_nC_r$. That is, in the combination symbol of the coefficient, the number of things chosen is the same as the index of x.

(4) Since

$$ {}_nC_0 = {}_nC_n,\ {}_nC_1 = {}_nC_{n-1},\ {}_nC_2 = {}_nC_{n-2},\ {}_nC_r = {}_nC_{n-r}, $$

it follows that the coefficients of terms equally distant from the beginning and the end are equal.

(5) If we substitute $-x$ for x, all the odd powers of x will be negative and the even powers will be positive.

$$ \therefore\ (a - x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \ldots $$

(6) Since every power of 1 is 1, it follows that

$$ (1 + x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \ldots + nx^{n-1} + x^n. $$

Ex. 1.—Expand $(a + x)^5$ and $(3a - 2b)^4$.

$$ \begin{aligned} (a + x)^5 &= a^5 + {}_5C_1a^4x + {}_5C_2a^3x^2 + {}_5C_3a^2x^3 + {}_5C_4ax^4 + {}_5C_5x^5 \\ &= a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5. \\ (3a - 2b)^4 &= (3a)^4 + {}_4C_1(3a)^3(-2b) + {}_4C_2(3a)^2(-2b)^2 \\ &\qquad + {}_4C_3(3a)(-2b)^3 + (-2b)^4 \\ &= 81a^4 - 216a^3b + 216a^2b^2 - 96ab^3 + 16b^4. \end{aligned} $$

Ex. 2.—Find the 4th term and the 9th term in $(a + b)^{20}$.

The 4th term contains b^3 and therefore its coefficient is ${}_{20}C_3$ or 1140.

$\therefore$ the 4th term $= 1140a^{17}b^3$.

The 9th term contains b^8 and its coefficient is ${}_{20}C_8$.

$\therefore$ the 9th term $= {}_{20}C_8a^{12}b^8 = \frac{\lfloor 20}{\lfloor 8 \ \lfloor 12}a^{12}b^8$.

Ex. 3.—Find the middle term of $(2x - y)^6$.

Since there will be 7 terms in the expansion the middle term will be the 4th term.

$\therefore$ the middle term $= {}_6C_3(2x)^3(-y)^3 = -160x^3y^3$.

EXERCISE 38

Find the expansions of the following :

1. $(a + x)^6$. **2.** $(a - b)^5$. **3.** $(a + 2x)^6$. **4.** $(1 + x^2)^4$.

5. $(2x - a^2)^4$. **6.** $(2 - x)^6$ **7.** $(x - \frac{1}{2})^4$. **8.** $(1 + 2x + x^2)^3$.

9. Expand $(x + a)^{12}$ to 4 terms and $(a - 2b)^8$ to 3 terms.

10. Simplify $(x + a)^4 + (x - a)^4$ and $(1 + x)^5 - (1 - x)^5$.

11. Simplify $(\sqrt{2}+1)^4+(\sqrt{2}-1)^4$ and $(\sqrt{a}+\sqrt{b})^5-(\sqrt{a}-\sqrt{b})^5$.

12. Find the 4th term of $(a + x)^7$, 3rd of $(3a - b)^6$, 8th of $(2x - y)^9$, 6th of $\left(2x^2 - \frac{1}{2x}\right)^{15}$.

13. Find the middle term in the expansions of

$$(x + x^2)^{12}, \left(x - \frac{1}{x}\right)^8, (2x - 3y)^6.$$

14. Find the two middle terms in $(a + x)^5$ and in $\left(2m - \frac{1}{2m}\right)^9$.

15. Find the term containing a^5 in $(x + a)^{10}$, x^8 in $(1 + 2x)^{20}$, b^5 in $(2b - a)^8$.

16. Expand $(1 + x)^5(1 - x^2)^3$ as far as the term involving x^4.

17. Expand $(1 + x + x^2)^3$ by writing $1 + x + x^2$ as a binomial in the form $(1 + x) + x^2$.

18. Find the first four terms, in ascending powers of x, in the expansion of $(1 - 3x + x^2)^5$, and the first three terms in $(1 - 2x)^8 \div (1 - x)^2$.

19. If $(1+x)^4=1+a_1x+a_2x^2 + \ldots$ and $(1+x)^5=1+b_1x+b_2x^2 + \ldots$ find the value of $1 + a_1b_1 + a_2b_2 + a_3b_3$.

20. Find the coefficient of x^3 in $(1 - x)^5(2 + x)^4$.

21. Find the coefficient of x^3 in $(1 + x - 6x^2)^4$.

101. The General Term. In the expansion of $(a + x)^n$, the term which contains x^r is ${}_nC_r a^{n-r}x^r$.

This term is called the **general term** and is the $(r + 1)$th term.

The general term in the expansion of $(a + x)^n$

$$= {}_nC_r a^{n-r}x^r = \frac{\lfloor n}{\lfloor r \;\; \lfloor n - r} a^{n-r}x^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r} a^{n-r}x^r.$$

Ex.—Write down the general term in $(2a - 3b)^p$.

The general term, being the $(r + 1)$th term, is the term which contains $-3b$ to the index r and therefore $2a$ to the index $p - r$.

$\therefore$ the general term $= {}_pC_r(2a)^{p-r} \cdot (-3b)^r$

$$= \frac{\lfloor p}{\lfloor r \;\; \lfloor p - r} \cdot 2^{p-r} \cdot 3^r \cdot (-1)^r \cdot a^{p-r}b^r.$$

Since it is not known whether r is even or odd, we cannot say whether $(-1)^r$ is equal to $+1$ or -1.

102. In the expansion of $(a + x)^n$ the number of the term containing any given power of a or x is at once known.

Thus, the term containing x^5 is the 6th term and therefore its coefficient is ${}_nC_5$; the term containing x^r is the $(r+1)$th and its coefficient is ${}_nC_r$; the term containing a^p contains x^{n-p} and is therefore the $(n-p+1)$th term.

In the expansion of $(1+x^2)^n$, the 3rd term does not contain x^2 but $(x^2)^2$ or x^4. Similarly, the term containing x^{20} is not the 21st term but the 11th term, as x^{20} will appear in the expansion in the form $(x^2)^{10}$.

Ex. **1.**—Find the coefficient of x^{25} in $\left(x^2+\frac{1}{x}\right)^{20}$.

Since it is not known directly in which term x^{25} will occur, we will suppose it occurs in the general term.

The $(r+1)$th term $= \frac{\lfloor 20}{\lfloor r \,\lfloor 20-r}(x^2)^{20-r}\left(\frac{1}{x}\right)^r$.

This term contains $x^{40-2r} \div x^r$ or x^{40-3r},

$\therefore \quad 40-3r=25, \qquad \therefore \quad r=5.$

Therefore the required term is the 6th and its coefficient is

$${}_{20}C_5 \text{ or } \frac{\lfloor 20}{\lfloor 5 \,\lfloor 15} \text{ or } 15504.$$

If it was required to find the coefficient of x^{24} in the same expansion we would have $40-3r=24$ and therefore $r=5\frac{1}{3}$. But it is evident that r must be a positive integer, so that we could conclude that x^{24} does not appear in the expansion of $\left(x^2+\frac{1}{x}\right)^{20}$, or that the coefficient of x^{24} is zero.

Ex. **2.**—Find the term independent of x in $\left(x+\frac{1}{x^3}\right)^{24}$.

As in the preceding find the term in which x^0 occurs. It will be found to be the 7th term and the coefficient is ${}_{24}C_6$.

EXERCISE 39

Find the general term in the expansion of:

1. $(1+x^3)^{15}$. **2.** $(1+x)^{n+1}$. **3.** $(a+a^2)^m$. **4.** $\left(x-\frac{1}{x}\right)^n$.

Find the coefficient of :

5. x^{10} in $(1+x)^{16}$. **6.** x^n in $(1+x)^{2n}$. **7.** x^6 in $(1-x^2)^{15}$.

8. Find the coefficients of x^6, x^0 and x^{-15} in $\left(x+\frac{1}{x^2}\right)^{12}$.

9. In which of these expansions is there a term which is independent of x : $\left(x+\frac{1}{x}\right)^{14}$, $\left(x+\frac{1}{x}\right)^{15}$, $\left(x^2+\frac{1}{x^3}\right)^9$, $\left(x^2+\frac{1}{x^3}\right)^{10}$?

10. In the expansion of $\left(x^2+\frac{1}{x}\right)^n$, show that there will be a term independent of x if n is a multiple of 3.

11. Find the coefficient of x^p in the expansion of $\left(x^2-\frac{1}{x^2}\right)^n$. Under what condition will x^p occur in this expansion ?

12. Prove that the coefficient of x^r in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of x^r and x^{r-1} in the expansion of $(1+x)^n$. (Compare with Art. 91.)

13. Show that the coefficient of x^n in $(1+x)^{2n}$ is twice the coefficient of x^n in $(1+x)^{2n-1}$.

14. Find the rth term from the end in the expansion of $(a+b)^n$.

15. Which term in the expansion of $\left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}+\frac{x^{\frac{2}{3}}}{x^{\frac{1}{2}}}\right)^{12}$ is independent of x ?

16. Find the condition that the pth and qth terms in the expansion of $(1+x)^n$ will have equal coefficients.

17. In the expansion of $(1+x)^{27}$, the coefficients of the $(r+3)$th and $(3r+2)$th terms are equal; find r.

18. If the rth and $(2r+1)$th terms in the expansion of $(1+x)^n$ have equal coefficients, find r in terms of n.

19. When n is a positive integer show that $(5+2\sqrt{6})^n+(5-2\sqrt{6})^n$ is equal to an even integer.

20. In the expansion of $(x+a)^n$, the 2nd, 3rd and 4th terms are 15, 90 and 270 respectively; find x, a and n.

21. Find the first two terms in the expansion of $(1-\cdot0003)^4$ and thus obtain the value of $(\cdot9997)^4$ to 4 decimal places.

22. Find the value of $(1\cdot004)^7$ to 4 decimal places.

103. Properties of the Binomial Coefficients.

$(a+x)^n = a^n + {}_nC_1 a^{n-1}x + {}_nC_2 a^{n-2}x^2 + \ldots {}_nC_r a^{n-r}x^r + \ldots + x^n.$

If we put $a = 1$, and write C_0, C_1, C_2 . . . for 1, ${}_nC_1$, ${}_nC_2$ . . .

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_rx^r + \ldots + C_nx^n.$

(1) If we put $x = 1$ we get

$$(1 + 1)^n \text{ or } 2^n = C_0 + C_1 + C_2 + \ldots + C_n.$$

$\therefore$ the sum of the coefficients in the expansion of $(1 + x)^n$ is 2^n. (Compare with Art. 92.)

(2) If we put $x = -1$ we get

$$(1 - 1)^n \text{ or } 0 = C_0 - C_1 + C_2 - C_3 + C_4 - C_5 + \ldots$$

$$\therefore \quad C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots$$

$\therefore$ the sum of the odd coefficients (that is, the odd-numbered coefficients) is equal to the sum of the even coefficients.

Since the sum of all the coefficients is 2^n, it follows that the sum of the odd coefficients or of the even coefficients is $2^n \div 2$ or 2^{n-1}.

(3) $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{n-1}x^{n-1} + C_nx^n.$

Also $(1+x)^n = C_n + C_{n-1}x + C_{n-2}x^2 + \ldots + C_1x^{n-1} + C_0x^n,$

since $C_n = C_0$, $C_{n-1} = C_1$, and so on.

The coefficient of x^n in the product of these two series

$= C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

= the coefficient of x^n in the product of $(1 + x)^n$ and $(1+x)^n$ or $(1 + x)^{2n}$

$$= \frac{\lfloor 2n}{\lfloor n \rfloor n}.$$

$\therefore$ the sum of the squares of the coefficients is $\dfrac{\lfloor 2n}{\lfloor n \rfloor n}$.

Compare with Ex. 30, page 146, and check the result by finding the sum of the squares of the actual coefficients in the expansion of $(1 + x)^5$.

Ex. 1.—If C_0, C_1, C_2 . . . C_n are the coefficients in the expansion of $(1 + x)^n$, find the sum of

(1) $C_1 + 2C_2 + 3C_3 + \ldots + nC_n$.

(2) $C_0 + 2C_1 + 3C_2 + \ldots + (n + 1)C_n$.

(1) The series

$$= n + \frac{2n(n-1)}{\lfloor 2} + \frac{3n(n-1)(n-2)}{\lfloor 3} + \ldots + n.$$

$$= n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{\lfloor 2} + \ldots + 1\right\}.$$

$$= n(1+1)^{n-1} = n \,.\, 2^{n-1}.$$

(2) The series

$$= (C_0 + C_1 + C_2 + \ldots + C_n) + (C_1 + 2C_2 + 3C_3 + \ldots + nC_n).$$

$$= 2^n + n \,.\, 2^{n-1} = 2^{n-1}(2 + n).$$

Ex. 2.—Prove that $C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + \ldots + \frac{1}{n+1}C_n = \frac{2^{n+1}-1}{n+1}$.

Let S be the sum of the series, then

$$S = 1 + \frac{n}{2} + \frac{n(n-1)}{3\lfloor 2} + \frac{n(n-1)(n-2)}{4\lfloor 3} + \ldots + \frac{1}{n+1}.$$

$$\therefore \quad S = 1 + \frac{n}{\lfloor 2} + \frac{n(n-1)}{\lfloor 3} + \frac{n(n-1)(n-2)}{\lfloor 4} + \ldots + \frac{1}{n+1}.$$

Multiply each side by $n + 1$, so as to have the same number of factors in the numerator and denominator of each term.

$$\therefore \quad S(n+1) = (n+1) + \frac{(n+1)n}{\lfloor 2} + \frac{(n+1)n(n-1)}{\lfloor 3} + \ldots + 1.$$

Add 1 to each side to make the series a complete expansion.

$$\therefore \quad S(n+1) + 1 = 1 + (n+1) + \frac{(n+1)n}{\lfloor 2} + \ldots + 1 = (1+1)^{n+1} = 2^{n+1}.$$

$$\therefore \qquad S = \frac{2^{n+1}-1}{n+1}.$$

EXERCISE 40

1. Find the sum of the coefficients in the expansion of $(1 + x)^4$, $(x + 1)^{10}$, $(x + 2y)^6$, $(2x + 3y)^n$, $(1 + x + x^2)^5$.

2. Find the sum of the odd coefficients in $(1 + x)^7$.

3. Find the sum of $1 + (n - 1) + \frac{(n-1)(n-2)}{1.2} + \ldots + 1$,

and of $1 + 2n + \frac{2n(2n-1)}{1.2} + \frac{2n(2n-1)(2n-2)}{1.2.3} + \ldots + 1$.

4. Show that

$$n + \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)}{1.2.3} + \ldots + n = 2(2^{n-1} - 1).$$

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$,

Prove that

5. $C_0 + 2C_1 + C_2 + 2C_3 + C_4 + 2C_5 + \ldots = 3 \cdot 2^{n-1}$.

6. $\frac{1}{2}(1 + x)^n + \frac{1}{2}(1 - x)^n = 1 + C_2x^2 + C_4x^4 + \ldots$

7. $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \frac{4C_4}{C_3} + \ldots + \frac{nC_n}{C_{n-1}} = \frac{n(n+1)}{2}$.

8. $C_1 - 2C_2 + 3C_3 - 4C_4 + \ldots + (-1)^{n-1}nC_n = 0$.

9. $C_0 - \frac{1}{2}C_1 + \frac{1}{3}C_2 - \frac{1}{4}C_3 + \ldots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$.

10. $C_0 + 3C_1 + 5C_2 + 7C_3 + \ldots = 2^n(n + 1)$.

11. $C_0C_1 + C_1C_2 + C_2C_3 + \ldots + C_{n-1}C_n = \frac{\lfloor 2n}{\lfloor n+1 \rfloor n-1}$.

12. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} \ldots + C_{n-r}C_n = \frac{\lfloor 2n}{\lfloor n+r \rfloor n-r}$.

13. $2C_0 + 2^2 \cdot \frac{C_1}{2} + 2^3 \cdot \frac{C_2}{3} + 2^4 \cdot \frac{C_3}{4} + \ldots = \frac{3^{n+1} - 1}{n+1}$.

104. The numerically greatest Term in $(a + x)^n$.

Let T_r be the rth term in the expansion of $(a + x)^n$.

Then

$$T_r = \frac{n(n-1)(n-2)\ldots(n-r+2)}{\lfloor r-1} \cdot a^{n-r+1}x^{r-1}.$$

And

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{\lfloor r} \cdot a^{n-r}x^r.$$

$$\therefore \quad T_{r+1} = T_r \times \frac{n-r+1}{r} \cdot \frac{x}{a}.$$

Since $$\frac{n-r+1}{r} \cdot \frac{x}{a} = \left(\frac{n+1}{r} - 1\right)\frac{x}{a},$$

it follows that $$\frac{n-r+1}{r} \cdot \frac{x}{a}$$

decreases as r increases, and therefore the terms in the expansion will continue to increase numerically only as long as this quantity is greater than 1.

That is, $T_{r+1} > T_r$ as long as $\frac{n-r+1}{r} \cdot \frac{x}{a} > 1$,

or " " $\frac{n+1}{r} - 1 > \frac{a}{x}$,

" " " $\frac{n+1}{r} > \frac{a}{x} + 1$,

" " " $\frac{n+1}{\frac{a}{x}+1} > r$.

Thus, in $(a + x)^{20}$ where $a = 2$ and $x = 3$,

$$T_{r+1} > T_r \text{ as long as } \frac{21}{\frac{2}{3}+1} > r \text{ or } \frac{63}{5} > r.$$

But the greatest value of r so that $\frac{63}{5} > r$ is 12.

Therefore the greatest term in the expansion of $(2 + 3)^{20}$ is the term in which $r = 12$, that is the 13th term.

Again in $(a + x)^{15}$ where $a = 3$ and $x = 5$,

$$T_{r+1} > T_r \text{ as long as } \frac{16}{\frac{3}{5} + 1} > r \text{ or } 10 > r.$$

Here when $r = 10$, $T_{r+1} = T_r$ or $T_{11} = T_{10}$. Therefore the 10th and 11th terms are equal and these two terms are greater than any other term.

Similarly, in the general case :

(1) If $\dfrac{n+1}{\dfrac{a}{x} + 1}$ is not an integer, and if we suppose the greatest integer in it to be p, then the greatest term will be the term in which $r = p$, that is, the $(p + 1)$th term.

(2) If $\dfrac{n+1}{\dfrac{a}{x} + 1}$ is an integer equal to p, then the pth and $(p + 1)$th terms are equal and they are greater than any other term.

Since we are finding the *numerically* greatest term, no account is taken of the fact that some of the terms may be negative.

If the greatest coefficient in the expansion of $(1 + x)^n$ is required, it can be found directly from Art. 95.

Ex.—Find the value of the numerically greatest term in the expansion of $(a - x)^{12}$ when $a = \frac{1}{2}$ and $x = \frac{1}{3}$.

$$T_{r+1} = T_r \times \frac{n - r + 1}{r} \cdot \frac{x}{a} = T_r \times \frac{13 - r}{r} \cdot \frac{2}{3} = T_r \times \frac{26 - 2r}{3r}.$$

$$\therefore \quad T_{r+1} > T_r \text{ as long as } \frac{26 - 2r}{3r} > 1 \text{ or } 26 > 5r.$$

$\therefore$ the numerically greatest term is the 6th.

$$\text{The 6th term} = \frac{\lfloor 12}{\lfloor 5 \rfloor 7} a^7(-x)^5 = -\frac{\lfloor 12}{\lfloor 5 \rfloor 7} \cdot \frac{1}{2^7} \cdot \frac{1}{3^5} = -\frac{11}{432}.$$

EXERCISE 41

Find which term is numerically the greatest in :

1. $(1 + x)^{30}$ when $x = 5$.

2. $(2 - x)^{19}$ when $x = 3$.

3. $(a + \frac{1}{2})^{10}$ when $a = 6$.

4. $(2a + 3b)^{16}$ when $a = 5$, $b = 2$.

5. $(3x - 5y)^n$ when $n = 10$, $x = 2y$.

6. Find the value of the greatest term in the expansion of (*a*) $(1 + 2x)^{14}$ when $x = \frac{1}{8}$, (*b*) $(3m + 4n)^p$ when $p = 7$, $m = \frac{1}{4}$, $n = 2$.

7. Which term has the greatest coefficient in the expansion of $(1 + x)^{24}$? Of $(1 + \frac{1}{2}x)^{15}$?

8. Show that the 3rd term in the expansion of $(1 + \cdot 08)^{35}$ is the greatest.

9. Find which terms in the expansion of $(1 + \frac{1}{10})^{98}$ are greater and which are less than the 11th term.

EXERCISE 42 (Miscellaneous Examples)

1. Find the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$.

2. Find the coefficient of x^{33} in $(ax - bx^2)^{20}$.

3. Simplify $(1 + \sqrt{1 - x^2})^5 + (1 - \sqrt{1 - x^2})^5$.

4. Reduce to its simplest form the 5th term of $\left(\frac{3a^2}{\sqrt[3]{b}} - \frac{a^{\frac{1}{2}}}{3b^2}\right)^7$.

5. Show that the coefficient of the middle term of $(1 + x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1 + x)^{2n-1}$.

6. Which term in $\left(\frac{2x^7}{\sqrt[3]{x}} - \frac{x^{\frac{1}{3}}}{x^3}\right)^{22}$ does not contain x?

7. Find the coefficient of x^9 in $\left(x^2 + \frac{1}{x^3}\right)^{12}$ and of x^4 in

$$(1 + x + x^2)(1 - x)^{10}.$$

8. Find which is the greatest term in the expansion of $(1 + \cdot 07)^{50}$, also in $(3 - 2x)^{13}$ when $x = 5$.

9. If x^r occurs in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^n$ show that $2n - r$ is a multiple of 5.

10. Find the coefficient of x^4 in $(1 + 3x - x^2)^5$.

11. If three successive coefficients in the expansion of $(1 + x)^n$ be 220, 495 and 792, find n.

12. Show by means of the identity $(1 + x)^{n+2} = (1 + x)^n(1 + x)^2$ that ${}_{n+2}C_r = {}_nC_r + 2{}_nC_{r-1} + {}_nC_{r-2}$.

13. If the coefficients of the 5th, 6th and 7th terms in the expansion of $(1 + x)^{2n}$ are in A.P., find n.

14. If the rth, $(r + 1)$th and $(r + 2)$th coefficients in $(1 + x)^n$ be in A.P., show that $n^2 - n(4r + 1) + 4r^2 = 2$.

15. If C_r be the coefficient of x^r in the expansion of $(1 - x)^{2n-1}$, prove that $C_{r-1} + C_{2n-r} = 0$.

16. If the coefficient of x^3 in the expansion of $(1 - x)^n$ be -1140, find n.

17. Show that the middle term in the expansion of $(1 + x)^{2n}$ may be expressed in the form $\dfrac{1.3.5 \ldots (2n - 1)}{\underline{|n}} . 2^n . x^n$.

18. Show that the difference of the coefficients of x^{r+1} and x^r in $(1 + x)^{n+1}$ is equal to the difference of the coefficients of x^{r+1} and x^{r-1} in $(1 + x)^n$.

19. Which term in $(3 + 2x)^{13}$ has the greatest coefficient?

20. If a be the sum of the odd terms and b the sum of the even terms in $(1 + x)^n$, show that $(1 - x^2)^n = a^2 - b^2$.

21. Show that

$$1 + {}_nC_1\left(\frac{x}{1-x}\right) + {}_nC_2\left(\frac{x}{1-x}\right)^2 + \ldots + {}_nC_n\left(\frac{x}{1-x}\right)^n = (1 - x)^{-n}.$$

22. Show that the general term in the expansion of

$$\left(1 + \frac{x}{m}\right)^m \text{ is } \left(1 - \frac{1}{m}\right)\left(1 - \frac{2}{m}\right)\left(1 - \frac{3}{m}\right) \ldots \left(1 - \frac{r-1}{m}\right)\frac{x^r}{\underline{|r}}.$$

23. Find the term independent of x in $\left(x^2 - 1 + \dfrac{1}{x^2}\right)^6$.

24. Show that

$$(1 - x^2)^n = (1 + x)^{2n} - 2nx(1 + x)^{2n-1} + \frac{2n(2n-2)}{\underline{|2}}x^2(1 + x)^{2n-2} - \ldots$$

25. If a_r is the coefficient of x^r in $(1 + x)^n$ and b_r of x^r in $(1 + x)^m$, find the sum of $a_0b_r + a_1b_{r-1} + a_2b_{r-2} + \ldots + a_rb_0$.

26. Show that the sum of the products two at a time of the coefficients in the expansion of $(1 + x)^n$ is $2^{2n-1} - \frac{1}{2} \cdot \frac{\lfloor 2n}{\lfloor n \lfloor n}$.

27. If C_0, C_1, $C_2 \ldots C_n$ are the coefficients in the expansion of $(1 + x)^n$, show that

$$C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n = \frac{\lfloor 2n}{\lfloor n - 2 \lfloor n + 2}.$$

28. Show that $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^n C_n^2 = 0$, if n be odd, and $= (-1)^{\frac{n}{2}} \frac{\lfloor n}{\lfloor \frac{n}{2} \lfloor \frac{n}{2}}$, if n be even.

29. Show that $1^2C_1 + 2^2C_2 + 3^2C_3 + \ldots + n^2C_n = n(n + 1)2^{n-2}$.

30. Show that

$$(C_0 + C_1)(C_1 + C_2) \ldots (C_{n-1} + C_n) = C_0C_1C_2 \ldots C_n \frac{(n + 1)^n}{\lfloor n}.$$

CHAPTER XIII

THE BINOMIAL THEOREM—ANY INDEX

105. When n is any positive integer we have shown that the expansion of $(1 + x)^n$ is

$$1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\lfloor r}x^r + \ldots$$

We will now examine whether this series is the expansion of $(1 + x)^n$ in some particular cases in which n is not a positive integer.

(1) If we put $n = -1$ the series becomes

$$1 + (-1)x + \frac{(-1)(-2)}{\lfloor 2}x^2 + \frac{(-1)(-2)(-3)}{\lfloor 3}x^3 + \ldots$$
$$= 1 - x + x^2 - x^3 + \ldots$$

But $(1 + x)^{-1} = \frac{1}{1 + x}$, and if we divide $1 + x$ into 1 the quotient in ascending powers of x is also $1 - x + x^2 - x^3 + \ldots$ and the process of division shows that the quotient is unending.

It would therefore appear that the expansion of $(1 + x)^n$, which we found when n was a positive integer, holds also when $n = -1$.

(2) If we put $n = -2$ the series becomes

$$1 + (-2)x + \frac{(-2)(-3)}{\lfloor 2}x^2 + \frac{(-2)(-3)(-4)}{\lfloor 3}x^3 + \ldots$$
$$= 1 - 2x + 3x^2 - 4x^3 + \ldots$$

But $(1 + x)^{-2} = \frac{1}{(1 + x)^2} = \frac{1}{1 + 2x + x^2}$, and if we divide $1 + 2x + x^2$ into 1 the quotient is also $1 - 2x + 3x^2 - 4x^3 + \ldots$ It would

therefore appear that the series is the expansion of $(1 + x)^n$ when $n = -2$.

(3) If we put $n = \frac{1}{2}$ the series becomes

$$1 + \tfrac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{\underline{|2}}x^2 + \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{\underline{|3}}x^3 + \ldots$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \ldots$$

But $(1 + x)^{\frac{1}{2}} = \sqrt{1 + x}$, and if we extract the square root of $1 + x$ in ascending powers of x the result is also $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots$ It would therefore appear that the series is the expansion of $(1 + x)^n$ when $n = \frac{1}{2}$.

In these cases we have found four terms only in each expansion. If we obtained more terms we would find that similar results would follow.

It thus appears that in some cases the binomial theorem is true for other than positive integral values of n. It may be proved that it is true for all values of n, but the proof is too difficult to be included in this text.

We will therefore assume that for all values of n, positive or negative, integral or fractional, the expansion of $(1 + x)^n$ is

$$1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}x^r + \ldots$$

We have seen that when n is a positive integer the series terminates by reason of a zero factor appearing in each term after the $(n + 1)$th term. But when n is not a positive integer it is impossible for a zero factor to appear in the coefficient of any term and consequently the series is an infinite one.

Ex. 1.—Expand $(1 + x)^{\frac{2}{3}}$ and $(1 - 2x)^{-\frac{3}{2}}$ each to four terms.

$$(1 + x)^{\frac{2}{3}} = 1 + \frac{2}{3}x + \frac{\frac{2}{3}\left(\frac{2}{3} - 1\right)}{\underline{|2}}x^2 + \frac{\frac{2}{3}\left(\frac{2}{3} - 1\right)\left(\frac{2}{3} - 2\right)}{\underline{|3}}x^3 + \ldots$$

$$= 1 + \frac{2}{3}x - \frac{1}{9}x^2 + \frac{4}{81}x^3 - \ldots$$

$$(1-2x)^{-\frac{3}{2}} = 1 + \left(-\frac{3}{2}\right)(-2x) + \frac{-\frac{3}{2}\left(-\frac{3}{2}-1\right)}{\underline{|2}}(-2x)^2$$

$$+ \frac{-\frac{3}{2}\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{\underline{|3}}(-2x)^3 + \ldots$$

$$= 1 + 3x + \frac{15}{2}x^2 + \frac{35}{2}x^3 + \ldots$$

Ex. 2.—Expand $(2a + 3x)^{-5}$ to four terms.

$$(2a+3x)^{-5} = \left\{2a\left(1+\frac{3x}{2a}\right)\right\}^{-5} = (2a)^{-5}\left(1+\frac{3x}{2a}\right)^{-5}$$

$$= \frac{1}{32a^5}\left\{1 + (-5)\frac{3x}{2a} + \frac{-5.-6}{\underline{|2}}\left(\frac{3x}{2a}\right)^2 + \frac{-5.-6.-7}{\underline{|3}}\left(\frac{3x}{2a}\right)^3 + \ldots\right\}$$

$$= \frac{1}{32a^5}\left(1 - \frac{15}{2}\cdot\frac{x}{a} + \frac{135}{4}\cdot\frac{x^2}{a^2} - \frac{945}{8}\cdot\frac{x^3}{a^3} + \ldots\right).$$

Ex. 3.—Find the 5th term in the expansion of $(1 + 3x)^{\frac{5}{3}}$ and the 6th term in the expansion of $(8 - 3x)^{\frac{2}{3}}$.

The 5th term in $(1+x)^n = \dfrac{n(n-1)(n-2)(n-3)}{\underline{|4}}x^4,$

$$\therefore \text{ the 5th term in } (1+3x)^{\frac{5}{3}} = \frac{\frac{5}{3}\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)\left(\frac{5}{3}-3\right)}{\underline{|4}}(3x)^4$$

$$= \frac{\frac{5}{3}\cdot\frac{2}{3}\cdot-\frac{1}{3}\cdot-\frac{4}{3}}{\underline{|4}}\cdot 3^4\cdot x^4 = \frac{5}{3}x^4.$$

The 6th term in $(8-3x)^{\frac{2}{3}}$ or in $8^{\frac{2}{3}}\left(1-\frac{3x}{8}\right)^{\frac{2}{3}}$

$$= 4\cdot\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\left(\frac{2}{3}-3\right)\left(\frac{2}{3}-4\right)}{\underline{|5}}\cdot\frac{3^5}{8^5}\cdot(-x)^5$$

$$= 4\cdot\frac{2.-1.-4.-7.-10}{\underline{|5}}\cdot\frac{-x^5}{8^5} = -\frac{7}{12288}x^5.$$

EXERCISE 43

Find the first four terms in the expansion of:

1. $(1 + x)^{\frac{5}{2}}$.
2. $(1 + x)^{-3}$.
3. $(1 + 2x)^{\frac{1}{2}}$.
4. $(1 + x^2)^{-2}$.
5. $(1 - x)^{-4}$.
6. $(2 - x)^{-4}$.
7. $(27 - 3x)^{\frac{1}{3}}$.
8. $(x + x^2)^{-5}$.
9. $\sqrt{1 - 4x}$.
10. $\sqrt[3]{1 + 3x}$.
11. $\sqrt[3]{1 - 2x + x^2}$.
12. $\sqrt[n]{a^n - nx}$.
13. Find the 4th term of $(1 + 2x)^{\frac{5}{2}}$.
14. 5th term of $(1 - \frac{3}{2}x)^{\frac{3}{2}}$.
15. 6th term of $(1 - x)^{-3}$.
16. 4th term of $(16 - 8x^2)^{\frac{1}{4}}$.
17. 10th term of $\sqrt{1 + 2x}$.
18. 7th term of $\dfrac{16}{\sqrt{1 - 2x}}$.
19. Show, by actual multiplication, that the first five terms in the product of the expansions of $(1 - x)^{-1}$ and $(1 - x)^{-2}$ are the first five terms in the expansion of $(1 - x)^{-3}$.
20. Find the value of x when the 4th term in the expansion of $(1 + x)^{\frac{9}{2}}$ is $\frac{1}{10}$.
21. Which is the first term with a negative coefficient in the expansion of $(1 + x)^{\frac{1}{3}}$? Of $(1 + x)^{\frac{3}{5}}$?
22. In the expansion of $(1 - 2x)^{\frac{3}{2}}$, there are only two terms with positive coefficients; which are they?
23. Which terms in $(1 - x)^{\frac{11}{3}}$ have negative coefficients?
24. Explain what signs the terms will have in the following expansions:

$$(1 - x)^{20},\ (1 - x)^{-20},\ (1 + x)^{\frac{2}{3}0},\ (1 - x)^{\frac{3}{5}},\ (1 - x)^{\frac{3}{7}}.$$

106. The General Term. In the expansion of $(1 + x)^n$, the $(r + 1)$th term or the general term is

$$\frac{n(n - 1)(n - 2) \ldots (n - r + 1)}{\lfloor r}x^r.$$

Ex. 1.—Find the general term in $\left(1 + \frac{3}{2}x\right)^{-\frac{3}{2}}$.

$$\text{The general term} = \frac{-\frac{2}{3}\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)\ldots\left(-\frac{2}{3}-r+1\right)}{\lfloor r}\left(\frac{3}{2}x\right)^r$$

$$= \frac{-\frac{2}{3}\cdot-\frac{5}{3}\cdot-\frac{8}{3}\ldots-\left(\frac{3r-1}{3}\right)}{\lfloor r}\cdot\frac{3^r}{2^r}\cdot x^r$$

$$= (-1)^r\cdot\frac{2\cdot5\cdot8\ldots(3r-1)}{2^r\lfloor r}x^r.$$

Ex. **2.**—Find the general term in $(1-x)^{-2}$.

$$\text{The general term} = \frac{-2(-2-1)(-2-2)\ldots(-2-r+1)}{\lfloor r}(-x)^r$$

$$= \frac{-2\cdot-3\cdot-4\ldots-(r+1)}{\lfloor r}(-1)^r\cdot x^r.$$

$$= (-1)^r\cdot\frac{2\cdot3\cdot4\ldots(r+1)}{\lfloor r}(-1)^r\cdot x^r.$$

But $(-1)^r\times(-1)^r=(-1)^{2r}=+1$ and $\frac{2\cdot3\cdot4\ldots(r+1)}{\lfloor r}=r+1$,

$\therefore$ the general term $=(r+1)x^r$.

Thus the 2nd term is $2x$, the 3rd is $3x^2$, the 4th is $4x^3$, and so on.

$\therefore\ (1-x)^{-2}=1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots$

Similarly, the general term of $(1-x)^{-1}$ is x^r,

$\therefore\ (1-x)^{-1}=1+x+x^2+x^3+\ldots+x^r+\ldots$

$$\text{The general term in } (1-x)^{-3}=\frac{-3\cdot-4\cdot-5\ldots(-3-r+1)}{\lfloor r}(-x)^r$$

$$=(-1)^{2r}\cdot\frac{3\cdot4\cdot5\ldots(r+1)(r+2)}{\lfloor r}x^r=\frac{(r+1)(r+2)}{2}x^r.$$

$$\therefore\ (1-x)^{-3}=1+3x+\frac{3\cdot4}{2}x^2+\frac{4\cdot5}{2}x^3+\ldots+\frac{(r+1)(r+2)}{2}x^r+\ldots$$

The expansions of $(1-x)^{-1}$, $(1-x)^{-2}$ and $(1-x)^{-3}$ are frequently used and it is advisable to commit them to memory.

Ex. 3.—Find the general term of $(1 - x)^{-n}$.

$$\text{The general term} = \frac{-n(-n-1)(-n-2)\dots(-n-r+1)}{\lfloor r}(-x)^r$$

$$= \frac{n(n+1)(n+2)\dots(n+r-1)}{\lfloor r}x^r.$$

$$\therefore \quad (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n+1)(n+2)}{\lfloor 3}x^3 + \dots$$

In this expansion it should be noted that the signs of the terms are all positive and that the factors in the coefficients increase by unity.

$$\text{Thus, } (1-x)^{-4} = 1 + 4x + \frac{4.5}{\lfloor 2}x^2 + \frac{4.5.6}{\lfloor 3}x^3 + \frac{4.5.6.7}{\lfloor 4}x^4 + \dots$$

$$= 1 + 4x + \frac{4.5}{\lfloor 2}x^2 + \frac{4.5.6}{\lfloor 3}x^3 + \frac{5.6.7}{\lfloor 3}x^4 + \frac{6.7.8}{\lfloor 3}x^5 + \dots$$

The three series in Ex. 2 are particular cases of $(1 - x)^{-n}$.

EXERCISE 44

Find the general term in the following expansions:

1. $(1 - x)^{-1}$. **2.** $(1 + x)^{-3}$. **3.** $(1 - 3x)^{-\frac{2}{3}}$.

4. $(a + bx)^{-2}$. **5.** $(1 - x)^{\frac{1}{2}}$. **6.** $(4 + x)^{\frac{3}{2}}$.

7. $\sqrt[3]{1 - 3x}$. **8.** $\dfrac{1}{\sqrt{1 + 4x}}$. **9.** $\sqrt[n]{a^n + nx}$.

10. Prove that the coefficient of x^m in the expansion of $(1 - x)^{-n-1}$ is equal to the coefficient of x^n in $(1 - x)^{-m-1}$.

11. Show that the $(n + 1)$th term in $(1 - x)^{-\frac{1}{2}}$ is the middle term in $\left(1 + \frac{x}{4}\right)^{2n}$.

12. Show that the coefficient of x^r in $(1 - 4x)^{-\frac{1}{2}}$ is $\dfrac{\lfloor 2r}{\lfloor r \lfloor r}$.

13. Show that, when n is a positive integer, the coefficients of x^{n-1} in $(1 - x)^{-n}$ and $(1 + x)^{2n-2}$ are equal.

14. Find the coefficient of x^r in $(1 + 2x + 3x^2 + 4x^3 + \ldots)^2$.

15. Show that $\frac{1}{2}m(m^2 + 11)$ is the coefficient of x^m in

$$(1 + x + 2x^2 + 3x^3 + 4x^4 + \ldots)^2.$$

16. Find the coefficient of x^r in $(1 - 2x + 3x^2 - 4x^3 + \ldots)^{-n}$.

107. Restriction on the Value of x in $(1 + x)^n$. In the expansion of $(1 + x)^n$, n may have any value, positive or negative, integral or fractional.

A particular case will show *that x may not have any value we please.*

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$$

If we put $x = 2$, then $(1 - x)^{-1} = (1 - 2)^{-1} = -1$, which is evidently not equal to $1 + 2 + 2^2 + 2^3 + \ldots$

Thus when $x = 2$ it is seen that $1 + x + x^2 + \ldots$ is not the arithmetical equivalent of $(1 - x)^{-1}$.

We have seen in the chapter on geometric progression (Art. 26) that this infinite series has a finite limit only when x is numerically less than unity. When x is less than unity the sum of the series is $\frac{1}{1 - x}$ or $(1 - x)^{-1}$.

We may therefore conclude that $1 + x + x^2 + \ldots$ is the arithmetical equivalent of $(1 - x)^{-1}$ only when x is numerically less than unity.

It may be shown, that when x is numerically less than unity that the arithmetical equivalent of $(1 + x)^n$ is

$$1 + nx + \frac{n \cdot n-1}{\lfloor 2} x^2 + \ldots \frac{n(n-1)(n-2) \ldots (n-r+1)}{\lfloor r} x^r + \ldots$$

The proof of this is beyond the scope of this text, and we will hereafter assume that in every binomial expansion, in which n is not a positive integer, there is this restriction upon the value of x.

Ex.—Expand $(a + x)^n$, (1) when $x < a$, (2) when $x > a$.

(1) $(a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n$, and since $\frac{x}{a} < 1$, the expansion

$$= a^n\left(1 + n \cdot \frac{x}{a} + \frac{n(n-1)}{\underline{|2}} \cdot \frac{x^2}{a^2} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}} \cdot \frac{x^r}{a^r} + \ldots\right).$$

(2) $(a + x)^n = x^n\left(1 + \frac{a}{x}\right)^n$, and since $\frac{a}{x} < 1$, the expansion

$$= x^n\left(1 + n \cdot \frac{a}{x} + \frac{n(n-1)}{\underline{|2}} \cdot \frac{a^2}{x^2} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}} \cdot \frac{a^r}{x^r} + \ldots\right).$$

108. Greatest Term in the Binomial Expansion. There are two standard forms of the binomial expansion.

$$(1)\quad (1 + x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}x^r + \ldots$$

$$(2)\quad (1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{\underline{|2}}x^2 + \ldots + \frac{n(n+1)\ldots(n+r-1)}{\underline{|r}}x^r + \ldots$$

When the index is positive, $T_{r+1} = T_r \times \frac{n - r + 1}{r}x$.

When the index is negative, $T_{r+1} = T_r \times \frac{n + r - 1}{r}x$.

Since the multiplier is $\left(\frac{n+1}{r} - 1\right)x$ or $\left(\frac{n-1}{r} + 1\right)x$, it is seen that the multiplier always decreases as r increases.

Ex. 1.—Find the numerically greatest term in the expansion of (1) $(1 + x)^{\frac{11}{2}}$ when $x = \frac{4}{5}$, (2) $(1 - x)^{-5}$ when $x = \frac{7}{10}$.

(1) $T_{r+1} = T_r \times \frac{n-r+1}{r}x = T_r \times \frac{\frac{11}{2}-r+1}{r} \cdot \frac{4}{5} = T_r \times \frac{26-4r}{5r}$.

$\therefore \quad T_{r+1} > T_r$ as long as $\frac{26-4r}{5r} > 1$ or $26 > 9r$.

The greatest value r can have so that $26 > 9r$ is 2.

$\therefore$ the 3rd term is the greatest.

(2) $T_{r+1} = T_r \times \frac{n+r-1}{r}x = T_r \times \frac{5+r-1}{r} \cdot \frac{7}{10} = T_r \times \frac{28+7r}{10r}$.

$\therefore \quad T_{r+1} > T_r$ as long as $\frac{28+7r}{10r} > 1$ or $28 > 3r$.

$\therefore$ the 10th term is the greatest.

Ex. 2.—Find the numerically greatest term in

$$(3x + 5y)^{1\frac{3}{4}} \text{ when } x = 7 \text{ and } y = 4.$$

Since $(3x + 5y)^{1\frac{3}{4}} = (3x)^{1\frac{3}{4}}\left(1 + \frac{5y}{3x}\right)^{1\frac{3}{4}}$, it is sufficient to find the greatest term in $\left(1 + \frac{5y}{3x}\right)^{1\frac{3}{4}}$.

$$T_{r+1} = T_r \times \frac{1\frac{3}{4}-r+1}{r} \cdot \frac{20}{21} = T_r \times \frac{85-20r}{21r}.$$

$\therefore \quad T_{r+1} > T_r$ as long as $85 - 20r > 21r$ or $85 > 41r$.

$\therefore$ the 3rd term is the greatest.

In the general case the greatest term may be found as in the preceding examples, similar to the method employed in Art. 104.

109. Summation of Series. Many binomial series may be summed by observing that they are particular cases of the standard formulæ stated in the preceding article.

Ex. 1.—Sum

$$1 + n\left(\frac{1-x}{1+x}\right) + \frac{n(n+1)}{\underline{|2}}\left(\frac{1-x}{1+x}\right) + \ldots \text{ to infinity.}$$

This is evidently a binomial expansion in which the index is negative since the factors in the coefficients increase. Also it is clear that $\frac{1-x}{1+x}$ takes the place of x in the standard series.

$$\therefore \text{ the series} = \left(1 - \frac{1-x}{1+x}\right)^{-n} = \left(\frac{2x}{1+x}\right)^{-n} = \frac{(1+x)^n}{2^n x^n}.$$

Ex. 2.—Show that

$$1 - \frac{1}{8} + \frac{1.3}{8.16} - \frac{1.3.5}{8.16.24} + \ldots = \frac{2}{\sqrt{5}}.$$

Here the index is also negative. Since the factors in the coefficients in the standard series increase by *unity*, the factors in the numerator of the 4th term must be changed to $\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}$ and the denominator must be $\underline{|3}$.

The 4th term then $= -\frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{\underline{|3}} \cdot \frac{2^3}{8^3} = -\frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{\underline{|3}} \left(\frac{1}{4}\right)^3$.

$$\therefore \text{ the series} = 1 - \frac{1}{2}\left(\frac{1}{4}\right) + \frac{\frac{1}{2} \cdot \frac{3}{2}}{\underline{|2}}\left(\frac{1}{4}\right)^2 - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{\underline{|3}}\left(\frac{1}{4}\right)^3 + \ldots$$

$$= \left(1 + \frac{1}{4}\right)^{-\frac{1}{2}} = \left(\frac{5}{4}\right)^{-\frac{1}{2}} = \left(\frac{4}{5}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}.$$

EXERCISE 45

Find the numerically greatest term in :

1. $(1 + x)^{\frac{5}{2}}$, when $x = \frac{2}{3}$. **2.** $(1 - x)^{-7}$, when $x = \frac{1}{2}$.

3. $(3 - 2x)^{\frac{1}{2}}$, when $x = \frac{1}{3}$. **4.** $(x - 2y)^n$, when $x = 5$, $y = 2$, $n = \frac{1}{2}$.

5. $(\frac{1}{2}x + \frac{1}{3}y)^{-4}$, when $\frac{x}{y} = \frac{7}{6}$. **6.** $(x + y)^{-n}$, when $\frac{y}{x} = \cdot 9$, $n = 12$.

7. Show that the greatest term in the expansion of $(1 + x)^{\frac{1}{2}}$, when $x = \frac{2}{3}$, is equal to $1\frac{1}{2}$.

8. Expand $\frac{1}{1+3x}$ as a series in ascending powers of x and also as a series in descending powers of x.

9. Expand $\left(\frac{1}{x} - \frac{1}{x^2}\right)^{-4}$, (1) when $x > 1$, (2) when $x < 1$.

10. Show that

$$1 + n\left(\frac{x}{1+2x}\right) + \frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2 + \ldots = \left(\frac{1+2x}{1+x}\right)^n.$$

11. Prove that $1 + \frac{1}{6} + \frac{1.3}{1.2}\cdot\frac{1}{6^2} + \frac{1.3.5}{1.2.3}\cdot\frac{1}{6^3} + \ldots = \sqrt{\frac{3}{2}}$.

12. Find the sum of $1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \ldots$, to 4 decimal places.

13. Show that $1 + \frac{1}{3^2} + \frac{1.4}{1.2}\cdot\frac{1}{3^4} + \frac{1.4.7}{1.2.3}\cdot\frac{1}{3^6} + \ldots = \sqrt[3]{1{\cdot}5}$.

14. Show that $1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \ldots$ is equal to

$$1 + \frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \ldots$$

15. Show that $1 + \frac{2n}{3} + \frac{2n(2n+2)}{3.6} + \frac{2n(2n+2)(2n+4)}{3.6.9} + \ldots$

$$= 2^n\left\{1 + \frac{n}{3} + \frac{n(n+1)}{3.6} + \frac{n(n+1)(n+2)}{3.6.9} + \ldots\right\}.$$

16. Show that $1 + \frac{1}{2}\left(\frac{3}{4}\right) + \frac{1.3}{2.4}\left(\frac{3}{4}\right)^2 + \frac{1.3.5}{2.4.6}\left(\frac{3}{4}\right)^3 + \ldots = 2$.

17. Show that $1 + \frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30} + \ldots = \sqrt[3]{\frac{10}{7}}$.

18. Show that $2 + \frac{5}{3\,|\,2} + \frac{5.7}{3^2.\,|\,3} + \frac{5.7.9}{3^3.\,|\,4} + \ldots = 3\sqrt{3}$.

19. Solve $1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots = \sqrt{1{\cdot}25}$.

20. Find the sum of the infinite series whose $(n+1)$th term is

$$\frac{(n+1)(n+2)(n+3)}{|\,3}\,\frac{1}{2^n}.$$

110. Ex. 1.—Find the coefficients of x^{10} and x^r in the expansion, in ascending powers of x, of $\frac{2+3x}{(1-x)^2}$.

$$\frac{2+3x}{(1-x)^2} = (2+3x)(1-x)^{-2}$$
$$= (2+3x)(1+2x+3x^2+\ldots+(r+1)x^r+\ldots).$$

In the product the only terms which will contain x^{10} are those obtained by multiplying 2 by $11x^{10}$ and $3x$ by $10x^9$.

$\therefore$ the coefficient of $x^{10} = 22 + 30 = 52$.

Similarly, the coefficient of $x^r = 2(r + 1) + 3r = 5r + 2$.

Ex. 2.—Find the coefficient of x^n in

$$(3 - 5x + 7x^2) \div (1 + x).$$

$$(3 - 5x + 7x^2)(1 + x)^{-1} = (3 - 5x + 7x^2)\{1 - (-x)\}^{-1}$$
$$= (3 - 5x + 7x^2)\{1 + (-x) + (-x)^2 + (-x)^3 + \ldots + (-x)^n + \ldots\}.$$

$\therefore$ the coefficient of $x^n = 3(-1)^n - 5(-1)^{n-1} + 7(-1)^{n-2}$
$= (-1)^{n-2}(3 + 5 + 7) = 15(-1)^{n-2}$
$= 15(-1)^n$, since $(-1)^{n-2} = (-1)^n$.

Ex. 3.—Find the sum of the coefficients of the first $(r + 1)$ terms in the expansion of $(1 - x)^{-n}$.

Let the coefficients be $a_0, a_1, a_2, a_3 \ldots$

Then $(1 - x)^{-n} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_rx^r + \ldots$

and $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$

In the product of these two series the coefficient of x^r
$= a_0 + a_1 + a_2 + \ldots + a_r$
= the coefficient of x^r in $(1 - x)^{-n} \times (1 - x)^{-1}$ or in $(1 - x)^{-n-1}$
$= \dfrac{(n + 1)(n + 2) \ldots (n + r)}{\lfloor r}$.

EXERCISE 46

Find the coefficients of x^{20} and x^r in:

1. $\dfrac{(1 + x)^2}{(1 - x)^2}$. **2.** $\dfrac{1 + x + x^2}{(1 - x)^3}$. **3.** $\dfrac{(2 - x)^2}{(x - x^2)^2}$. **4.** $\dfrac{1 + x - 2x^2}{(1 + x)^2}$.

5. Find the first three terms in the expansions of

$$\frac{1}{(1 + x)^2(1 - x)}, \quad \frac{1 + 2x - x^2}{(1 + x)^3}, \quad \frac{\sqrt{1 + 4x} + \sqrt{1 + 2x}}{\sqrt[3]{1 + 3x}}.$$

6. Find the coefficient of x^4 in $(1 - \frac{1}{2}x)^8(1 - x)^{-1}$.

7. Find the coefficients of x^{100} and x^r in $\dfrac{2+x}{(1+x)^3}$.

8. Show that

$$(2 + 3x + 4x^2) \div (1 - x)^2 = 2 + 7x + 16x^2 + \ldots + (9r - 2)x^r + \ldots$$

9. Show that the coefficient of x^{n+r} in the expansion of $(1 + nx^n)(1 - x)^{-2}$ is $(1 + n)(1 + r) + n$.

10. Expand $(x - 1)^2 \div \sqrt{1 + x}$ to four terms in ascending powers of x.

11. In the expansion of $\dfrac{5 - 6x}{(1 - x)^2}$, which is the only power of x whose coefficient is zero? Which in $\dfrac{5 - 6x}{(1 - x)^3}$?

12. By expressing $\dfrac{1 + 2x}{1 + x + x^2}$ in the form $\dfrac{(1 + 2x)(1 - x)}{1 - x^3}$, find the coefficients of x^{3m}, x^{3m-1}, x^{3m-2}.

13. Find the coefficient of x^{99} in $(1 + x + x^2)^{-2}$.

14. Find the coefficients of x^{20}, x^{21}, x^{22} in $\dfrac{1}{1 + x + x^2 + x^3}$.

15. Find the sum of the first $n + 1$ coefficients in the expansion of $(1 - x)^{-2}$ and verify by summing $1 + 2 + 3 + \ldots$ to $(n + 1)$ terms.

16. Show that the sum of the first $n + 1$ coefficients in the expansion of $(1 - x)^{-n}$ is equal to the coefficient of x^n in $(1 + x)^{2n}$.

17. When m is a positive integer find the sum of $m + 1$ terms of the series

$$1 + m + \frac{m(m+1)}{1 \cdot 2} + \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3} + \ldots$$

111. Approximations. In the expansion of $(1 + x)^n$, when x is small compared with unity, a sufficient approximation to $(1 + x)^n$ may be found by taking only a small number of terms.

The first approximation to $(1 + x)^n$ is $1 + nx$ and the second approximation is $1 + nx + \dfrac{n(n-1)}{2}x^2$, and so on.

Thus, to $(1{\cdot}002)^{10}$ or $(1 + {\cdot}002)^{10}$ the first approximation is $1 + 10({\cdot}002)$ or $1{\cdot}02$.

The second approximation is

$$1 + 10(\cdot002) + 45(\cdot002)^2 \text{ or } 1\cdot02018.$$

Ex. 1.—Find an approximation to $\sqrt{1\cdot005}$ or $(1\cdot005)^{\frac{1}{2}}$.

$$(1 + \cdot005)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(\cdot005) + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{\lfloor 2}(\cdot005)^2 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{\lfloor 3}(\cdot005)^3 + \ldots$$

$$= 1 + \cdot0025 - \cdot000003125 + \cdot0000000078 - \ldots$$

Here it is seen that the terms decrease very rapidly. If we consider only two terms the result is 1·0025, which is correct to 4 decimal places as the third term begins with 5 ciphers. Similarly, if we take three terms the result will be correct to 7 decimal places.

Ex. 2.—Find to 5 decimal places the cube root of 994.

The nearest perfect cube to 994 is 1000. We therefore change 994 into $1000 - 6$.

$$\sqrt[3]{994} = (1000 - 6)^{\frac{1}{3}} = 1000^{\frac{1}{3}}(1 - \cdot006)^{\frac{1}{3}} = 10(1 - \cdot006)^{\frac{1}{3}}$$

$$= 10\left\{1 - \tfrac{1}{3}(\cdot006) + \frac{\frac{1}{3} \cdot -\frac{2}{3}}{\lfloor 2}(\cdot006)^2 - \frac{\frac{1}{3} \cdot -\frac{2}{3} \cdot -\frac{5}{3}}{\lfloor 3}(\cdot006)^3 + \ldots\right\}$$

$$= 10\,(1 - \cdot002 - \cdot000004 - \cdot00000001 - \ldots)$$

$$= 10 - \cdot02 - \cdot00004 - \cdot0000001 - \ldots$$

$$= 9\cdot97996, \text{ by taking the first three terms only.}$$

Ex. 3.—Find the first approximation to $\dfrac{(8 + 3x)^{\frac{2}{3}}}{(2 + 3x)\sqrt{4 - 5x}}$.

The expression $= (8 + 3x)^{\frac{2}{3}}(2 + 3x)^{-1}(4 - 5x)^{-\frac{1}{2}}$

$= 4(1 + \frac{3}{8}x)^{\frac{2}{3}} \cdot \frac{1}{2}(1 + \frac{3}{2}x)^{-1} \cdot \frac{1}{2}(1 - \frac{5}{4}x)^{-\frac{1}{2}}$

$= (1 + \frac{1}{4}x)(1 - \frac{3}{2}x)(1 + \frac{5}{8}x)$, neglecting x^2, x^3, etc.

$= (1 - \frac{5}{4}x)(1 + \frac{5}{8}x)$, neglecting x^2

$= 1 - \frac{5}{8}x$, neglecting x^2

112. Homogeneous Products. To find the Number of Homogeneous Products of r Dimensions which can be formed from

n Letters $a, b, c \ldots$, each Letter being taken any Number of Times.

If we multiply together n infinite series like

$$1 + ax + a^2x^2 + a^3x^3 + \ldots$$
$$1 + bx + b^2x^2 + b^3x^3 + \ldots$$
$$1 + cx + c^2x^2 + c^3x^3 + \ldots$$

the product in ascending powers of x is

$$1+x\,(a+b+c+\ldots)+x^2(a^2+b^2+\ldots+ab+ac+\ldots)+\ldots$$

The coefficient of x^r will be the *sum* of all terms of r dimensions which can be formed from the n letters $a, b, c \ldots$, each letter being taken any number of times from 0 to r.

Since we require only the *number* of terms in the coefficient of x^r in the product, we may let $a, b, c \ldots$ each equal 1, then the coefficient of x^r will be the number of products required.

If $a = b = c = \ldots = 1$, each of the given series is $(1 - x)^{-1}$ and their product is $(1 - x)^{-n}$.

$\therefore$ the number of homogeneous products is the coefficient of x^r in the expansion of $(1 - x)^{-n}$

$$= \frac{n(n+1)(n+2)\ldots(n+r-1)}{\lfloor r} = \frac{\lfloor n+r-1}{\lfloor r \;\; \lfloor n-1},$$

since n is a positive integer.

Ex. 1.—Find the number of combinations of n things taken r at a time, where repetitions are allowed.

This is evidently the same as the number of products, each of r dimensions, which can be formed from n letters and their powers.

$\therefore$ the required number $= \dfrac{\lfloor n+r-1}{\lfloor r \;\; \lfloor n-1}$ or ${}_{n+r-1}C_r$.

Ex. 2.—Find the number of terms in the expansion of $(a + b + c)^{10}$.

In the expansion each term will be of 10 dimensions, formed from the three letters a, b, c and their powers.

$$\therefore \text{ the number of terms} = \frac{\lfloor 3 + 10 - 1}{\lfloor 10 \ \lfloor 3 - 1} = \frac{\lfloor 12}{\lfloor 10 \ \lfloor 2} = 66.$$

Ex. 3.—Find the coefficient of x^3y^2z in the expansion of $(x + y + z)^6$.

The result is the same as the number of ways in which 6 letters may be written, of which 3 are x's, 2 are y's and 1 is z.

$$\therefore \text{ the coefficient} = \frac{\lfloor 6}{\lfloor 3 \ \lfloor 2} = 60.$$

EXERCISE 47

Find the value of

1. $\sqrt[3]{1005}$ to 4 decimal places.
2. $(\cdot 998)^6$ to 5 places.
3. $(624)^{-\frac{3}{4}}$ to 7 places.
4. $\sqrt[5]{35 \cdot 2}$ to 4 places.
5. Show that $\sqrt{101} = 10 \cdot 0498756 \ldots$
6. Since $\frac{1 \cdot 003}{\cdot 997} = 1 \cdot 003(1 - \cdot 003)^{-1}$, show that $\frac{1 \cdot 003}{\cdot 997} = 1 \cdot 006$ approximately.
7. Find the cube root of 67 to the nearest thousandth.
8. Show that the ratio of 999^4 to 997^4 is nearly the same as the ratio of 1005 to 997.
9. Find the value of $\$(1 \cdot 03)^{15}$ to the nearest cent.
10. Find to the nearest integer the value of $1{,}000{,}000(1 \cdot 02)^{10}(\cdot 98)^{10}$.
11. The population of a town of 10,000 inhabitants increased at the rate of 5% per annum for 5 years. During the next 5 years it decreased at the same annual rate. Use the binomial theorem to find the population at the end of the 10 years.

12. When the temperature of a cube of silver whose edge is 1 inch is increased from 0° C. to 1° C., each edge is increased by ·0000192 of an inch. Find the approximate increase in the volume.

13. If the coefficient of cubical expansion of brass is ·0000567, find the coefficient of linear expansion.

14. The volume of a cube is 1·00042 cu. in. Find, to 5 decimal places, the area of one of its faces.

When x is so small that its square and higher powers may be neglected, find the value of :

15. $(1 + x)^{10}(1 - x)^3$.

16. $(1 + 5x)^9 - (1 + 9x)^5$.

17. $\dfrac{(1 + x)^{\frac{2}{3}}(1 - 2x)^{\frac{1}{2}}}{1 - \frac{5}{6}x}$.

18. $\dfrac{(1 + x)^{\frac{1}{2}} + (1 - x)^{\frac{1}{2}}}{(1 - x)^{\frac{1}{2}} + (1 + x)^{\frac{1}{2}}}$.

19. Find the first approximation to $(1 + x) \div (1 - 3x)^5$.

20. Find the second approximation to $\dfrac{1}{(1 - 3x)^2(1 + 2x)^3}$.

21. How many terms are there in the expansion of $(a + b + c + d)^8$, and of $(a_1 + a_2 + \ldots + a_r)^n$?

22. How many combinations are there of 20 things taken 4 at a time, when each thing may be repeated any number of times ?

23. In how many ways can a boy select a dozen marbles in a shop where there are 5 kinds for sale ?

24. Find the coefficient of a^3b^2 and of $abcd^2$ in the expansion of $(a + b + c + d)^5$.

EXERCISE 48 (Miscellaneous Examples)

1. Show that $x^n = 1 + n\left(1 - \dfrac{1}{x}\right) + \dfrac{n(n + 1)}{1 \cdot 2}\left(1 - \dfrac{1}{x}\right)^2 + \ldots$

2. Find the value of $\sqrt[4]{623}$ to the nearest thousandth.

3. Find the coefficient of x^r in the expansion of

$$\frac{4 + 2x + 3x^2}{1 - 2x + x^2}.$$

4. Find the coefficient of x^n in $(1 + x)(1 - x)^{-4}$.

5. Show that $\sqrt{2} = \frac{10}{7}(1 - \cdot 02)^{\frac{1}{2}}$ and thus find the square root of 2 to 5 decimal places.

6. Show that $\frac{a^2b^2}{(a+b)^4} = \frac{a^2}{b^2} - \frac{4}{1}\cdot\frac{a^3}{b^3} + \frac{4\,.\,5}{1\,.\,2}\cdot\frac{a^4}{b^4} - \frac{4\,.\,5\,.\,6}{1\,.\,2\,.\,3}\cdot\frac{a^5}{b^5} + \ldots,$ if b is numerically greater than a.

7. Expand $(1 + 3x) \div (1 - \frac{1}{2}x)^{\frac{1}{2}}$ to the second power of x.

8. Find the value of the greatest term in the expansion of $(x + 4a)^8$, when $x = \frac{1}{2}$ and $a = \frac{1}{3}$.

9. Show that the coefficient of x^n in $\frac{(1+x)^n}{1-x}$ is 2^n.

10. If $\frac{x^2}{y^2} = \frac{\cdot 998}{\cdot 992}$, show that x exceeds y by about $\cdot 3\%$.

11. Sum $1 + \frac{1}{10} + \frac{1\,.\,3}{10\,.\,20} + \frac{1\,.\,3\,.\,5}{10\,.\,20\,.\,30} + \ldots$ to infinity.

12. Find the coefficient of $x^5y^7z^{13}$ in $(x + y + z)^{25}$.

13. Show that

$$(1+x)^{2n} = (1+x)^n + nx(1+x)^{n-1} + \frac{n(n+1)}{2}x^2(1+x)^{n-2} + \ldots$$

14. If the coefficient of linear expansion of copper is $\cdot 000017$, find the coefficient of cubical expansion.

15. Show that

$$1 - \frac{n+x}{1+x} + \frac{(n+2x)(n-1)}{\underline{|2}\ (1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{\underline{|3}\ (1+x)^3} + \ldots = 0.$$

16. Show that $\frac{1}{2} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{4} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{4^2} + \ldots = \frac{8}{\sqrt{3}} - 4.$

17. If c be a quantity so small that c^3 may be neglected in comparison with l^3, find an approximate value of $\sqrt{\frac{l}{l+c}} + \sqrt{\frac{l}{l-c}}$ free of surds, in terms of c and l.

18. Find the coefficient of x^n in the expansion of $\frac{3-5x}{1-3x+2x^2}$, by expressing it in the form $\frac{2}{1-x} + \frac{1}{1-2x}$.

19. Show that the coefficient of x^{2n} in the expansion of $\frac{1}{(1-x)^3(1+x)^4}$ is $\frac{1}{6}(n+1)(n+2)(n+3)$.

20. Find the remainder after n terms in the expansion of $(1 - x)^{-1}$ and in the expansion of $(1 - x)^{-2}$.

21. If a, b, c, d be any four consecutive coefficients in the expansion of $(1 + x)^n$, show that $\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}$.

22. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, then
$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$.

23. In the expansion of $\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$ show that the coefficients of the $(2n - 1)$th and $(2n)$th terms are equal.

24. When n is any positive integer :

(a) Show that $(2 - \sqrt{3})^n$ = a proper fraction.

(b) Show that $(2 + \sqrt{3})^n + (2 - \sqrt{3})^n$ = an even integer.

(c) If $(2 - \sqrt{3})^n = F$, and $(2 + \sqrt{3})^n = I + F_1$ where I is an integer and F_1 is a proper fraction, show that $F + F_1 = 1$ and that I is an odd integer.

25. Show that the integral part of $(8 + 3\sqrt{7})^n$ is odd when n is a positive integer.

26. If I be the integral part and F the fractional part of $(3\sqrt{3} + 5)^{2n+1}$, then $F(I + F) = 2^{2n+1}$.

27. Find the coefficient of x^n in the expansion of $\frac{(1+x)^n}{(1-x)^3}$, by writing the fraction in the form $\{2 - (1 - x)\}^n(1 - x)^{-3}$.

28. When n is a positive integer, find the sum of the first $n + r$ coefficients in the expansion of $\frac{(1+x)^n}{1-x}$.

29. Find the sum of the first n terms of the series

$$1 + \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{3}{4} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} + \tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{5}{6} \cdot \tfrac{7}{8} + \ldots$$

30. A candidate is examined on three papers on each of which the maximum is 100 marks. Find the number of ways in which he can obtain a total of 99 marks.

CHAPTER XIV

ANNUITIES

THE solutions of the problems which appear in this chapter depend chiefly upon the arithmetical theory of compound interest. For the purpose of review a few examples in compound interest are first given.

113. Amount of a Sum at Compound Interest. When \$1 is loaned at 5% the interest being compounded annually, the amount at the end of 1 year is $\$1 \times 1{\cdot}05$, at the end of 2 years is $\$1 \times (1{\cdot}05)^2$, at the end of 10 years is $\$1 \times (1{\cdot}05)^{10}$, and so on.

If the rate is 5% per annum compounded half-yearly, the amount at the end of the first half-year is $\$1 \times 1{\cdot}025$, at the end of 1 year is $\$1 \times (1{\cdot}025)^2$, at the end of 10 years is $\$1 \times (1{\cdot}025)^{20}$, and so on.

Ex. 1.—Find the amount of \$250 in 10 years at 4% compounded yearly.

The amount of \$1 in 10 years $= \$1 \times (1{\cdot}04)^{10}$,

$\therefore$ the amount of \$250 in 10 years $= \$250 \times (1{\cdot}04)^{10}$.

If this result is to be simplified, the value of $(1{\cdot}04)^{10}$ may be found from the interest tables or by means of logarithms.

Since $(1{\cdot}04)^{10} = 1{\cdot}48024$,

$\therefore$ the amount $= \$250 \times 1{\cdot}48024 = \$370{\cdot}06$.

$\therefore$ the interest $= \$370{\cdot}06 - \$250 = \$120{\cdot}06$.

Ex. 2.—Find the amount of \$132 in $6\frac{1}{2}$ years at 5% per annum, compounded half-yearly.

Here there are 13 interest periods. At the end of each period $2\frac{1}{2}$% is added.

$\therefore$ the amount $= \$132(1{\cdot}025)^{13} = \$132 \times 1{\cdot}37851 = \$181{\cdot}96.$

Ex. 3.—Find the interest on \$500 in $5\frac{1}{2}$ years at 5% per annum compounded yearly.

The amount at the end of 5 years is $\$500 \times (1{\cdot}05)^5$.
For the additional half year $2\frac{1}{2}$% is added.

$\therefore$ the amount in $5\frac{1}{2}$ years $= \$500(1{\cdot}05)^5(1{\cdot}025),$
$= \$654{\cdot}09.$

$\therefore$ the interest $= \$154{\cdot}09.$

Ex. 4.—In what time will \$150 amount to \$330 at 6% compounded yearly?

Let n years be the required time,

$$\therefore \quad \$150(1{\cdot}06)^n = \$330,$$

$$\therefore \quad (1{\cdot}06)^n = \frac{330}{150} = 2{\cdot}2,$$

$$\therefore \quad n \log 1{\cdot}06 = \log 2{\cdot}2,$$

$$\therefore \quad n = \frac{\log 2{\cdot}2}{\log 1{\cdot}06} = \frac{{\cdot}3424}{{\cdot}0253} = 13{\cdot}5 +$$

Therefore the time required is $13\frac{1}{2}$ years approximately.

114. Present Value and Discount. If a person owes me \$100 due in 5 years, and if he wishes to pay the debt now, he should pay me such a sum as would amount to \$100 when loaned at the current rate of interest.

If the rate of interest is 6% and if $\$x$ is the sum he should pay, then the amount of $\$x$ in 5 years at 6% should be \$100.

$$\therefore \quad \$x \times (1{\cdot}06)^5 = \$100,$$

$$\therefore \quad x = \frac{100}{1{\cdot}06^5} = 100 \times \frac{1}{1{\cdot}06^5} = 74{\cdot}73.$$

Therefore the present value of \$100 due in 5 years when money is worth 6% is $\frac{\$100}{(1{\cdot}06)^5}$ or \$74·73. Similarly, the present value of

\$1 due in 10 years at 3% = \$1 ÷ $(1{\cdot}03)^{10}$,
\$150 due in 4 years at $2\frac{1}{2}$% = \$150 ÷ $(1{\cdot}025)^4$,
\$75·50 due in $2\frac{1}{2}$ years at 6% = \$75·50 ÷ $\{(1{\cdot}06)^2 \times 1{\cdot}03\}$.

It should be noted that the amount of a sum is found by *multiplication* and the present value is found by *division*.

The difference between a sum due at a future time and its present value is called the **discount.**

Thus, the discount on \$500 due in 8 years, if money is worth 4%, is \$500 − $\frac{\$500}{(1{\cdot}04)^8}$ or \$134·66.

EXERCISE 49

1. State, in the unsimplified form, the amount of :

(*a*) \$125 in 7 years at 3%.
(*b*) \$200 in 4 years at $5\frac{1}{2}$%.
(*c*) \$720 in $4\frac{1}{2}$ years at 4%, compounded half-yearly.
(*d*) \$800 in $6\frac{1}{4}$ years at 6%, compounded half-yearly.

2. State the present value of :

(*a*) \$1000 due in 5 years, money worth 5%.
(*b*) \$360 due in $4\frac{1}{2}$ years, money worth 6%.
(*c*) \$820 due in $5\frac{1}{2}$ years, money worth $4\frac{1}{2}$% per annum, payable half-yearly.

3. Find the interest on \$620 left unpaid for 6 years, the rate being 5% compounded half-yearly.

4. Find the present value of \$400 due at the end of 30 years, money being worth 4%.

5. Find the amount of \$100 in 10 years at 6%, compounded (1) yearly, (2) half-yearly, (3) quarterly.

6. What is the discount on \$1200 due in $5\frac{1}{2}$ years, if money is worth 5% compounded half-yearly ?

7. In what time will any sum of money double itself at compound interest at 5% ? At 3% ?

8. In what time will \$1 amount to \$2·30 at 4% ?

9. In what time will \$560 amount to \$1500 at 5% ?

10. What rate per cent. compounded yearly is equivalent to 2% per quarter, compounded quarterly ?

11. Prove that the amount of a sum of money at compound interest for 10 years at $2\frac{1}{2}\%$ is greater than the amount for 5 years at 5%, in each case compounded yearly.

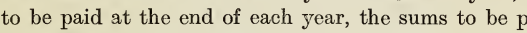
12. If the rate of interest is such that a sum of money doubles itself in 10 years, find the amount of \$1 in 100 years at the same rate.

13. For how long should \$100 be left to accumulate at 5% in order to amount to double the accumulated value of \$100 deposited for 8 years at 3% ?

14. A man owes \$800 due in 3 years and \$1200 due in 8 years, without interest. He desires to exchange these obligations for a contract calling for two equal payments, without interest, one due in 2 years and the other in 4 years. If money is worth 6%, what should each of the equal payments be ?

115. Annuity. When a fixed sum is payable yearly, or at other regular intervals, it is called an **annuity.**

If A rents a farm from B for 6 years at \$200 a year, the rent to be paid at the end of each year, the sums to be paid by A and the times at which they are payable might be represented graphically as in this diagram.



On this line the length of each space is one year, the point A being the present time.

If money is worth 5% and the rent is left unpaid until the end of the 6 years, the amount then due on account of the first year's rent is $\$200(1{\cdot}05)^5$, on account of the second $\$200(1{\cdot}05)^4$, and so on.

The total amount, in dollars, due at the end of the 6 years is

$$200(1{\cdot}05)^5 + 200(1{\cdot}05)^4 + 200(1{\cdot}05)^3 + 200(1{\cdot}05)^2 + 200(1{\cdot}05) + 200.$$

To find the sum of this series we might simplify each term in order and add the results.

It may be found more readily by recognizing that it is a geometric series.

Since the sum of

$$a + ar + ar^2 + \ldots \text{ to } n \text{ terms} = a\frac{r^n - 1}{r - 1},$$

then the sum of $200 + 200(1{\cdot}05) + \ldots$ to 6 terms

$$= 200 \cdot \frac{1{\cdot}05^6 - 1}{1{\cdot}05 - 1} = \frac{200}{{\cdot}05}(1{\cdot}05^6 - 1) = 4000\{1{\cdot}05^6 - 1\}$$

$$= 4000 \times {\cdot}34010 = 1360{\cdot}40,$$

$\therefore$ the amount at the end of 6 years is \$1360·40.

If we are required to find the sum which A should pay now, if he wishes to pay the 6 years' rent in advance, we may do so by expressing the present value of each year's rent and finding their sum.

The present value of \$200 due in 1 year is $\frac{\$200}{1{\cdot}05}$, of \$200 due in 2 years is $\frac{\$200}{1{\cdot}05^2}$, and so on.

The present value, in dollars, of the 6 years' rent

$$= \frac{200}{1{\cdot}05} + \frac{200}{1{\cdot}05^2} + \frac{200}{1{\cdot}05^3} + \frac{200}{1{\cdot}05^4} + \frac{200}{1{\cdot}05^5} + \frac{200}{1{\cdot}05^6}.$$

Considering the first term as $\frac{200}{1{\cdot}05^6}$, since

$a = \frac{200}{1{\cdot}05^6}$, $r = 1{\cdot}05$, $n = 6$, the present value

$$= \frac{\$200}{1{\cdot}05^6} \cdot \frac{1{\cdot}05^6 - 1}{1{\cdot}05 - 1} = \$4000 \cdot \frac{1{\cdot}05^6 - 1}{1{\cdot}05^6}$$

$$= \$4000\left(1 - \frac{1}{1{\cdot}05^6}\right) = \$4000(1 - {\cdot}74622) = \$1015{\cdot}12.$$

We have here found the amount of an annuity of \$200 payable at the end of each year for 6 years, when the rate is 5%, to be \$1360·40 and the present value of the same annuity to be \$1015·12.

The present value might also have been found by dividing the amount at the end of 6 years by $1{\cdot}05^6$.

In the same way the present value or the amount of any annuity may be found.

The steps in the method are as follows :

(1) Make a diagram showing what sums are payable and when they are to be paid.

(2) Indicate the amount or present value of each sum and write the total as a series.

(3) Find the sum of this series and simplify by using logarithms or interest tables.

Ex.—A man deposits \$100 at the end of each year in a bank which pays 3% interest, compounded half-yearly. Find the total amount of his deposits when the 20th has been made.

The amount, in dollars, beginning with the last

$$= 100 + 100(1{\cdot}015)^2 + 100(1{\cdot}015)^4 + \ldots + 100(1{\cdot}015)^{38},$$

$$= 100 \cdot \frac{1{\cdot}015^{40} - 1}{1{\cdot}015^2 - 1} = 100 \cdot \frac{{\cdot}81402}{{\cdot}03023} = 2693.$$

the amount is \$2693, to the nearest dollar.

116. Deferred Annuity. An annuity which does not begin at once is said to be **deferred**. When an annuity is deferred 5 years, the first payment is due at the end of 6 years.

Ex.—Find the present value of an annuity of \$60 deferred 4 years and to run 15 years, if money is worth 6%.

The present value in dollars

$$= \frac{60}{1{\cdot}06^5} + \frac{60}{1{\cdot}06^6} + \ldots + \frac{60}{1{\cdot}06^{19}} = \frac{60}{1{\cdot}06^{19}}\left(\frac{1{\cdot}06^{15} - 1}{1{\cdot}06 - 1}\right)$$

$$= 1000\left(\frac{1{\cdot}06^{15} - 1}{1{\cdot}06^{19}}\right) = 1000\left(\frac{1}{1{\cdot}06^4} - \frac{1}{1{\cdot}06^{19}}\right) = 1000(\cdot 79209 - \cdot 33051)$$

$$= 461{\cdot}58.$$

$\therefore$ the present value is \$461·58.

117. Perpetuity or Perpetual Annuity. When an annuity continues for ever it is called a **perpetuity** or a **perpetual annuity.**

Thus, the rent of land or the interest on money is a perpetual annuity.

Ex. **1.**—A university gives a scholarship of \$100 every year for proficiency in mathematics. If money is worth 5%, find the present value of all these future payments.

The present value, in dollars,

$$= \frac{100}{1{\cdot}05} + \frac{100}{1{\cdot}05^2} + \frac{100}{1{\cdot}05^3} + \ldots \text{ to inf.}$$

This is an infinite G.P. in which $a = \frac{100}{1{\cdot}05}$ and $r = \frac{1}{1{\cdot}05}$, and since the sum of $a + ar + ar^2 \ldots$ to infinity $= \frac{a}{1 - r}$, when $r < 1$ (Art. 26),

$$\therefore \text{ the present value} = \frac{\frac{100}{1{\cdot}05}}{1 - \frac{1}{1{\cdot}05}} = \frac{100}{1{\cdot}05 - 1} = \frac{100}{\cdot 05} = 2000.$$

The present value therefore is \$2000.

This result might be obtained by finding the sum on which the interest for each year at 5% is \$100, and that sum is evidently \$2000.

Ex. **2.**—Find the present value of a perpetual annuity of \$80 deferred 6 years, money worth 4%.

Here the first payment is due at the end of 7 years,

$$\therefore \text{ the present value} = \frac{\$80}{1{\cdot}04^7} + \frac{\$80}{1{\cdot}04^8} + \ldots \text{ to inf.}$$

$$= \frac{\$80}{1{\cdot}04^7} \div \left(1 - \frac{1}{1{\cdot}04}\right) = \frac{\$80}{1{\cdot}04^6 \times {\cdot}04} = \$1580{\cdot}62.$$

Or, we might find the value of the annuity at the time that it begins, which would be $\$80 \div {\cdot}04$, and then the present value would be obtained by dividing this result by $1{\cdot}04^6$.

EXERCISE 50

1. Find the amount of an annuity of \$150 to continue for 12 years, if money is worth 5%.

2. I deposit \$100 in a bank at the beginning of each year. If the bank pays 4% per annum, compounded yearly, find what sum the bank will owe me at the end of 10 years.

3. Find the present value of an annuity of \$240 to continue 15 years, the rate being 6%.

4. What is the present value of a perpetual annuity of \$250 if money is worth 5% ?

5. A man spends \$50 a year more than his income. Allowing compound interest at 3%, find the amount of this accumulated debt in 25 years.

6. A minor receives on account of his estate \$1200 on his 16th and each succeeding birthday. If interest is at $3\frac{1}{2}$%, what sum should be charged against his estate after he receives the payment on his 21st birthday ?

7. A man pays \$150 at the beginning of each year for 15 years for an endowment policy of \$2500. Find the accumulated value of the payments, the rate of interest being 6%.

8. What sum should be paid now for an annuity of \$280 deferred 5 years and to continue 10 years, money being worth $3\frac{1}{2}$% ?

9. What sum invested now at 5% will provide, at the end of 4 years, for a perpetual annuity of \$200 ?

10. A man wishes to provide for his son an annuity of \$700, the first payment to be made 5 years from now and the annuity to run for 9 payments. If money is worth 4%, find what sum he should pay for the annuity.

11. *A* owns an annuity of \$50 to begin now and to continue for ever, and *B* owns one of \$100 to begin in 4 years and to continue for 20 years. Compare their present values, money being worth 5%.

12. A man aged 54 years, in receipt of a pension of \$100 a year, wishes to commute it for a present payment. If his expectation of life is 17 years and interest is reckoned at 5%, how much should he receive?

13. If money is worth 5%, what was the value on January 1, 1920, of an annuity of \$100 payable on every January 1 from 1930 to 1939 inclusive?

14. A man leaves property worth \$750 a year to his wife during her life, and after her death it is to go to his son during his life. If the wife's expectancy of life is 12 years and the son's is 34 years, find the present value of the son's annuity, money being worth 6%.

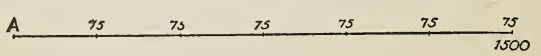
15. Find the present value of a perpetual annuity of \$10 payable at the end of the first year, \$20 at the end of the second year, and so on, increasing \$10 each year, the rate of interest being 5%.

16. A person paid a premium of \$43·20 at the beginning of each year for 20 payments and at the end of 20 years received an endowment of \$1227·60. Show that he received a higher rate than 3% on his investment.

118. Present Value of a Mortgage, Bond or Debenture. The payments to be made on a note, mortgage, bond or debenture differ from an ordinary annuity, in that the face of the loan is repayable at the end of the time as well as the interest during the time that it runs.

Ex. 1.—I own a mortgage for \$1500 bearing interest at 5% payable annually and having 6 years to run. If money is worth 6%, for what sum should the mortgage sell?

Here there are payments on account of interest of \$75 at the end of each year and at the end of 6 years the \$1500 is also paid.



The present value, in dollars,

$$= \frac{75}{1{\cdot}06} + \frac{75}{1{\cdot}06^2} + \cdots \frac{75}{1{\cdot}06^6} + \frac{1500}{1{\cdot}06^6} = \frac{75}{1{\cdot}06^6} \cdot \frac{1{\cdot}06^6 - 1}{1{\cdot}06 - 1} + \frac{1500}{1{\cdot}06^6}$$

$$= \frac{75}{{\cdot}06}\left(\frac{1{\cdot}06^6 - 1}{1{\cdot}06^6}\right) + \frac{1500}{1{\cdot}06^6} = 1250\left(1 - \frac{1}{1{\cdot}06^6}\right) + \frac{1500}{1{\cdot}06^6}$$

$$= 1250 + \frac{250}{1{\cdot}06^6} = 1250 + 176{\cdot}24 = 1426{\cdot}24.$$

The selling price therefore should be \$1426·24.

Ex. 2.—A \$1000 bond bearing interest at 6% per annum payable half-yearly has 15 years to run and is repayable at a premium of 5%. Find its present value if money is worth 5% compounded half-yearly.

On this bond there are 30 half-yearly payments of interest each of \$30, and at the end of 15 years \$1050 is due.

The present value, in dollars,

$$= \frac{30}{1{\cdot}025} + \frac{30}{1{\cdot}025^2} + \cdots + \frac{30}{1{\cdot}025^{30}} + \frac{1050}{1{\cdot}025^{30}}.$$

$$= \frac{30}{1{\cdot}025^{30}}\left(\frac{1{\cdot}025^{30} - 1}{1{\cdot}025 - 1}\right) + \frac{1050}{1{\cdot}025^{30}} = 1200 - \frac{150}{1{\cdot}025^{30}} = 1128{\cdot}49$$

$\therefore$ the present value is \$1128·49.

Ex. 3.—A town borrows \$6000 at 5%. They agree to repay the loan, both principal and interest, in 10 equal annual payments. What is the annual payment?

If we suppose the annual payment to be \$$x$, then the amount of an annuity of \$$x$ for 10 years is equal to the amount of \$6000 for 10 years.

$$\therefore \quad x(1{\cdot}05)^9 + x(1{\cdot}05)^8 + \ldots + x(1{\cdot}05) + x = 6000(1{\cdot}05)^{10}.$$

$$\therefore \quad x \cdot \frac{1{\cdot}05^{10} - 1}{{\cdot}05} = 6000(1{\cdot}05)^{10}, \qquad \therefore \quad x = \frac{300(1{\cdot}05)^{10}}{1{\cdot}05^{10} - 1} = 777{\cdot}03.$$

$\therefore$ the annual payment is \$777·03.

119. Sinking Fund. When a municipality borrows money they frequently repay the loan by collecting a fixed sum in

the taxes for each year. Part of this sum pays the annual interest due on the loan and the balance when invested amounts to sufficient to repay the loan when it is due. This balance is called a **sinking fund.**

Ex.—A city borrows \$30,000 at 4% for 30 years. What sum must be collected in the taxes each year to pay the interest and provide a sinking fund at 5% sufficient to repay the loan when due?

Here the annual interest is \$1200, and if we suppose \$$x$ in addition to be collected, then the amount of an annuity of \$$x$ for 30 years calculated at 5% must be equal to \$30,000.

$\therefore\ x(1{\cdot}05)^{29} + x(1{\cdot}05)^{28} + \ldots + x(1{\cdot}05) + x = 30{,}000.$

Solving we find $x = 451{\cdot}54$.

$\therefore$ the sum to be collected $= \$1200 + \$451{\cdot}54 = \$1651{\cdot}54$.

120. Depreciation Funds. Every kind of physical property, such as machines, buildings, etc., depreciates with use. Even when kept in good repair there will come a time when these assets must be renewed. The business man should therefore make provision for the replacement of property that will be destroyed through use.

A fund set aside to replace depreciable property when it is worn out is called a **depreciation fund.**

Although an asset may be so impaired by use that it cannot longer be used for the purpose for which it was intended, it may still have a **scrap value** amounting to a small fraction of its cost. Thus the scrap value of a steam engine must be at least equal to the value of the old iron in it.

Ex.—What sum must be set aside annually to provide for replacing an auto-truck costing \$2000 and having a probable life of 5 years, if the depreciation fund can be invested at 4%, and if the scrap value is \$50?

If $\$x$ is the annual payment to the depreciation fund, then the amount of an annuity of $\$x$ a year for 5 years must be \$1950.

$$\therefore \quad x(1{\cdot}04^4 + 1{\cdot}04^3 + 1{\cdot}04^2 + 1{\cdot}04 + 1) = 1950,$$

$$\therefore \quad x \cdot \frac{1{\cdot}04^5 - 1}{{\cdot}04} = 1950,$$

$$\therefore \quad x = 360{\cdot}03.$$

The annual payment therefore is \$360·03.

EXERCISE 51

1. I hold a mortgage for \$4000 payable in 6 years and bearing interest at 6% per annum payable yearly. What is the cash value of the mortgage if the current rate of interest is 5% ?

2. A mortgage for \$2500 bearing interest at 8% per annum payable half-yearly has 4 years to run, the first payment of interest being due in 6 months. Find its present value if money is worth 6% per annum payable half-yearly.

3. At what price should City of Toronto 10-year bonds sell if money is worth 6% and if the bonds are payable at par and bear interest at 5% yearly ?

4. An industrial company issues bonds bearing interest at 5% payable yearly and redeemable in 20 years at a premium of 10%. If I wish to make 6% on my money, what price can I afford to pay for each \$100 bond ?

5. A debenture bearing interest at 6% per annum payable half-yearly has 10 years to run, when it is repayable at a premium of 20% Find its cash value if money is worth 5% per annum compounded half-yearly

6. A town borrows \$12,000 at 5%. If it is to be repaid in 20 equal annual instalments, find the annual payment.

7. Find the annual payment into a sinking fund to be accumulated at 4%, established for the purpose of extinguishing a debt of \$25,000 due in 20 years.

8. If money is worth 6% per annum, payable half-yearly, what is the difference between the present value of a victory bond of \$1000, bearing interest at $5\frac{1}{2}$% payable half-yearly, (*a*) when it is repayable at par in 20 years, (*b*) when repayable in 10 years ?

9. A municipality borrows \$40,000 agreeing to pay interest thereon at $4\frac{1}{2}\%$ per annum. What amount of taxes must they raise each year to pay the interest and to provide a sinking fund which, being invested at 4% per annum, will cancel the debt in 30 years?

10. At the beginning of each year for the last 20 years a man paid \$84·50 to a life insurance company for an endowment policy of \$2000, which he now receives. How much more did he pay each year than would have been sufficient, if deposited in a bank at 4% per annum, to amount to the face of the policy?

11. A man pays \$240 yearly in advance for a 15-year endowment policy of \$4000. If money is worth 6% per annum, how much is he paying each year for the life risk?

12. A debt of \$20,000 is to be paid in 20 years by means of a sinking fund established at the time the debt is incurred. If the sinking fund can be accumulated at 4%, how large will the fund be at the end of 10 years?

13. What would be the annual depreciation charge for a steam locomotive costing \$12,000, having a probable life of 25 years and a scrap value of \$1000, if the depreciation fund can be accumulated at 5%?

14. On a 4% basis, find the annual charge for depreciation for a plant consisting of four parts, with costs, scrap values and probable lives as given in the following table:

Part	Cost	Scrap Value	Life
Building	\$11,000	\$1000	40 years
Engine	5,400	400	25 years
Boiler	2,150	150	15 years
Dynamo	6,300	300	18 years

15. Show that an annual payment of \$2309·75 for 5 years will pay off a loan of \$10,000 with interest at 5% annually.

16. A city with an assessed valuation of \$30,000,000 borrows \$300,000 at 6% to erect a municipal building. If principal and interest are to be repaid in 30 equal annual instalments, find how much the rate of taxation will be increased.

17. If money is worth 6%, show that a 4% bond having 3 years to run is a good investment at 94, if the security is satisfactory.

18. A bond dealer offers Province of Alberta 20-year bonds, bearing interest at 6% per annum payable half-yearly, at 97·72. He states that these bonds, if bought at this price, will pay the purchaser 6·20% on his investment. Verify the correctness of this statement.

121. General Formulæ for Annuities. The problems so far considered in this chapter have been arithmetical in character. We might make them general by using symbols to represent the terms employed.

In theoretical work it is customary to use a rate per unit instead of a rate per cent.

Thus, if the rate is i per unit, the interest on \$1 is \$$i$. If the rate was 5% then i, in this case, would be ·05.

The amount of 1 in 1 year is $1 + i$.

The amount of 1 in 2 years is $(1 + i)^2$.

The amount of 1 in n years is $(1 + i)^n$.

If we suppose P to be the principal then the amount of P in n years is $P(1 + i)^n$.

For brevity we will suppose the amount of 1 in 1 year to be a. Then $1 + i = a$.

$\therefore$ the amount of P in n years $= Pa^n$.

Similarly, the present value of \$$P$ due at the end of n years, where i is the rate per unit

$$= \frac{\$P}{(1 + i)^n} = \frac{\$P}{a^n} = \$P \times a^{-n}.$$

122. Amount and Present Value of an Annuity. Let P be the annual payment, n the number of years, i the rate per unit, and $1 + i = a$, then the amount

$$= P + Pa + Pa^2 + Pa^3 + \ldots Pa^{n-1},$$

$$= P \cdot \frac{a^n - 1}{a - 1} = \frac{P}{i}(a^n - 1).$$

The present value is evidently the amount divided by a^n.

$\therefore$ the present value $= \frac{P}{i}\left(\frac{a^n - 1}{a^n}\right) = \frac{P}{i}(1 - a^{-n}).$

123. Present Value of a Perpetuity. In the preceding article if n is indefinitely large then a^{-n} is indefinitely small.

$\therefore$ the present value of a perpetuity $= \frac{P}{i}$.

Or the present value is the sum to infinity of

$$\frac{P}{a} + \frac{P}{a^2} + \frac{P}{a^3} + \dots,$$

which is $\frac{P}{a-1}$ or $\frac{P}{i}$.

124. Present Value of a Deferred Annuity. Suppose P to be the annual payment and that the annuity is deferred m years and then runs n years, the present value

$$= \frac{P}{a^{m+1}} + \frac{P}{a^{m+2}} + \dots \frac{P}{a^{m+n}},$$

$$= \frac{P}{a^{m+n}} \cdot \frac{a^n - 1}{a - 1} = \frac{P}{i}\left(\frac{a^n - 1}{a^{m+n}}\right) = \frac{P}{i}(a^{-m} - a^{-m-n}).$$

Here, if n is indefinitely large, a^{-m-n} is indefinitely small, and the present value is $\frac{P}{i}a^{-m}$.

Therefore the present value of a perpetuity deferred m years is $\frac{P}{i}a^{-m}$.

EXERCISE 52

1. At what rate % per annum compounded half-yearly will a sum of money double itself in 14 years?

2. What rate % per annum compounded quarterly is equivalent to 8% per annum compounded annually?

3. What is the discount on $\$P$ due in n years where i is the rate per unit per annum?

4. In how many years will \$100 amount to \$252 at 5% compounded yearly?

5. Find the present value of an annuity of \$150 to run for n years, the rate of interest being 3%, and interpret the result when n becomes indefinitely large.

6. Find the present value of an annuity of P per annum to continue for n years, where i is the rate per unit per annum compounded half-yearly.

7. If A is the present value of an annuity to continue for n years and $A + B$ is its value for $2n$ years, find the annual payment.

8. What annuity is equivalent to the sum of \$200 paid at the end of every two years, the rate being 5%?

9. A man spends \$20 each year on tobacco. If these sums were deposited in a bank at the end of each year, in how many years would they amount to \$1000 if the bank pays 4% compounded half-yearly?

10. A person has a capital of \$20,000, for which he receives interest at 4%. If he withdraws from interest at the beginning of each year \$1500 for expenses, in how many years will he have spent all his capital?

11. A deposit of P dollars a year is to be made for a period of 20 years into a fund to be used at the end of that period to purchase an annuity of A dollars a year for 10 years. Derive a formula for P in terms of A, the interest rate in each case being 3%.

12. A village built a school house costing \$12,000 and raised \$1720 a year to pay for it. Allowing interest at 6%, how many whole years will it require to cancel the debt and what will be the balance then due?

13. Show that the amount of a sum of money at simple interest is a first approximation to the amount at compound interest.

14. The compound interest on a certain sum for the second year is \$49·92 and for the third year is \$51·9168. Find the sum and the rate %.

15. A company borrows money at r% per annum interest payable yearly, and invests the money in mortgages at r% per annum interest payable m times a year. Find the gain in 1 year on P dollars borrowed and invested, assuming interest received to be invested immediately.

16. Find the present value of an infinite number of yearly payments of P dollars each, if a dollars is the present value of b dollars due in one year.

17. 20 years ago a man insured his life for \$1000, paying an annual premium of \$30. During the first 10 years money could have been invested at 6% and during the last 10 years at 4%. If he should die now, which would have been the more profitable investment for his family?

18. Express the present value of an annuity of 1 per annum payable half-yearly for n years, the first payment being 6 months hence, the rate of interest being i per unit per annum, compounded half-yearly.

(a) What does this expression become when n is infinite?

(b) What does it become when n is finite and $i = 0$?

19. Show that every terminating annuity is the difference of two perpetual annuities. Use this method to find the present value of an annuity of \$50 deferred 6 years and to run 20 years, if money is worth 5%.

20. A man saves \$360 a year, depositing his savings at the beginning of each quarter where he will receive interest compounded quarterly at 4% per annum. Find his accumulated savings 5 years from the date of the first deposit. (Use the binomial theorem in evaluating the power of the interest fraction.)

21. The annual subscription to a certain club is \$50 (payable at the beginning of each year), but a member can secure a life membership by making a single payment of \$1000. If he is satisfied with 3% per annum for his money, how long must he live to make it worth his while to take a life membership?

CHAPTER XV

PROBABILITY AND INSURANCE

125. In this chapter will be found some simple theorems in probability and the application of probability to elementary work in insurance.

Before giving a general definition of probability some simple illustrations are first discussed.

(1) When a coin is thrown it may fall either head or tail and the two are equally likely to happen, so that we say that the chance or probability of its falling head is 1 to 2 or $\frac{1}{2}$, and the probability of its falling tail is also $\frac{1}{2}$.

(2) If a bag contains 2 white balls and 3 black balls and 1 ball is drawn at random, any one of the 5 balls may be drawn and all are equally likely to be drawn. We say that the chance or probability of a white ball being drawn is 2 to 5 or $\frac{2}{5}$ and that a black ball will be drawn is 3 to 5 or $\frac{3}{5}$.

(3) If an ordinary die, with its six faces numbered from 1 to 6, be thrown, the number on the upper face may be 1, 2, 3, 4, 5 or 6, and all are equally likely to appear. That is, there are six possible ways in which the die may fall. Then we say that the probability that a particular number, say 5, may be thrown is $\frac{1}{6}$ and the probability that 5 is not thrown is $\frac{5}{6}$. If we wish to know the probability that the throw will be either 1 or 2, out of the six possible cases there are two which are favourable, and we say that the probability that the throw will be 1 or 2 is $\frac{2}{6}$ or $\frac{1}{3}$.

126. Probability. The Probability that, among several equally likely Events, a given Event will happen is the Ratio of the Number of favourable Cases to the whole Number of possible Cases.

Thus, in (1) of the preceding article, if we wish to state the probability that the coin will fall heads, there is one case which is favourable out of two possible cases, and therefore the probability is $\frac{1}{2}$.

In (2), if we wish the probability of drawing a black ball, there are three favourable cases out of five possible cases, and therefore the probability is $\frac{3}{5}$.

In (3), if we wish the probability of throwing more than 2, there are four favourable cases out of six possible cases, and therefore the probability is $\frac{4}{6}$ or $\frac{2}{3}$.

If a is the number of favourable cases and b the number of unfavourable cases, then the probability of the event happening is $\frac{a}{a+b}$ and of its failing is $\frac{b}{a+b}$.

Ex. 1.—Of 100,000 children of age 10 it is estimated that 749 will die within a year. What is the probability that a child of 10 will live a year?

Here out of 100,000 cases there are 99,251 which are favourable, and therefore the probability is $\frac{99,251}{100,000}$ or ·99251.

Ex. 2.—Two coins are thrown into the air. What is the probability that both will fall heads?

Here there are four possibilities. The coins may both fall heads, or both tails, or the first one head and the other tail, or the first tail and the other head. There is therefore one favourable case out of four equally possible cases, and the required probability is $\frac{1}{4}$.

Ex. 3.—A committee of two persons is to be selected from ten persons, what is the probability that two particular persons A and B will be selected?

The total number of ways of selecting two persons from ten is ${}_{10}C_2$ or 45. Therefore the probability that A and B will be selected is $\frac{1}{45}$.

EXERCISE 53

1. If a die is thrown, what is the probability that the number on the upper face will be 4? That it will be more than 4? That it will be less than 4?

2. A person holds one ticket in a lottery where there are 6 prizes and 24 blanks. What is the probability that he will win a prize? That he will not win a prize?

3. If two coins are thrown, what is the probability that they will both fall tails?

4. Of 100,000 persons living at age 10, statistics show that 57,917 will be alive at age 60. What is the probability that a boy of 10 will live to be 60? Will die before he is 60?

5. A bag contains 11 white balls of which 7 are marked with crosses and 4 are not, also 8 black balls, 3 with and 5 without crosses. If one ball is drawn at random, what is the probability that it will be (*a*) a white ball with a cross, (*b*) a black ball with a cross, (*c*) a ball without a cross?

6. If 12 balls, numbered 1 to 12, are placed in a bag and 2 are drawn at random, what is the probability that the 2 are numbered 3 and 10?

7. If 3 persons are chosen from 10, what is the probability that *A*, *B* and *C* will all be chosen? What is the probability that they will not all be chosen? What is the probability that none of them will be chosen?

8. From 20 tickets marked with the first 20 numbers one is drawn at random; find the chance that it is a multiple of 3 or of 7.

9. If 12 boys stand in a row, show that the probability that two particular boys will be together is 1 to 6.

10. The letters of the word *absurd* are placed in a row. Find the probability that the two vowels come together.

11. If p is the probability that an event will happen and q that it will fail, show that $p + q = 1$.

12. From a bag containing 4 white and 5 black balls 3 are drawn at random. What is the probability that the 3 will all be black?

127. Total Probability. If an event can happen in two or more different ways which are mutually exclusive, the probability that it will happen is the sum of the probabilities that it will happen in these different ways.

Thus, if the probability that A will win a race is $\frac{1}{3}$ and that B will win is $\frac{1}{4}$, then the probability that A or B will win it is $\frac{1}{3} + \frac{1}{4}$ or $\frac{7}{12}$, as the events are mutually exclusive.

128. Compound Probability. Two events are said to be independent if the happening of either one has no influence upon the other.

Thus, if a die is thrown twice the result of either throw has no influence upon the result of the other and therefore these are independent events.

Compound probability is the probability that two or more independent events will happen simultaneously or in succession.

Thus, if a die is thrown once the probability of the throw being 5 is $\frac{1}{6}$. If it is thrown a second time the probability of the throw being 5 is also $\frac{1}{6}$. If we wish to know the probability that *both* throws will be a 5 we have an example of compound probability. There are evidently 6×6 or 36 possible ways of throwing the die twice, of which only one will give the desired result.

Therefore the probability of throwing a 5 twice is $\frac{1}{36}$, which is seen to be the product of the partial probabilities.

We will therefore assume that the following theorem is true in all cases :

The compound probability of two or more independent events is the product of the probabilities of the events taken singly.

Ex. 1.—If a coin is thrown three times, what is the probability that each throw will be head ?

Here there are three independent events, the probability of each one singly being $\frac{1}{2}$. Therefore the probability that all of the three events will happen is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{8}$.

Ex. 2.—The probability that A can solve a problem is $\frac{1}{5}$ and that B can solve it is $\frac{1}{4}$; what is the probability that it will be solved if both work at it ?

The probability that A will fail to solve it is $1 - \frac{1}{5}$ or $\frac{4}{5}$, and that B will fail is $\frac{3}{4}$. These are independent events, therefore the probability that neither one will solve it is $\frac{4}{5} \times \frac{3}{4}$ or $\frac{3}{5}$.

Therefore the probability that one at least will solve it is $1 - \frac{3}{5}$ or $\frac{2}{5}$.

Or, we might reason directly thus:

(1) The probability that both will solve it is $\frac{1}{5} \times \frac{1}{4}$ or $\frac{1}{20}$.

(2) The probability that A will solve it and B will fail is $\frac{1}{5}(1 - \frac{1}{4})$ or $\frac{3}{20}$.

(3) The probability that B will solve it and A will fail is $\frac{1}{4}(1 - \frac{1}{5})$ or $\frac{1}{5}$

These are mutually exclusive events, therefore the probability that the problem will be solved is

$$\frac{1}{20} + \frac{3}{20} + \frac{1}{5} \text{ or } \frac{8}{20}.$$

EXERCISE 54

1. When two dice are thrown, what is the probability that one throw will be a 3 and the other a 4?

2. If a coin is thrown three times, what is the probability that it will fall heads and tails alternately?

3. What is the chance of throwing a 2 in the first only of two successive throws with an ordinary die?

4. If a die is thrown twice, what is the probability that the first throw will be less than 3 and the second more than 3?

5. A, B and C run in a race with other competitors. The probability that A will win is $\frac{1}{6}$, that B will win is $\frac{1}{5}$ and that C will win is $\frac{1}{4}$. What is the probability that none of the three will win? What is the probability that one of them will win?

6. If the probability that A will live 30 years is $\frac{2}{3}$ and that B will live 30 years is $\frac{1}{3}$, find the probability that (a) both will be alive in 30 years, (b) both will be dead, (c) A will be alive and B will be dead, (d) one at least will be alive.

7. The probability that a certain event will happen is $\frac{2}{7}$ and that another event, independent of the former, will happen is $\frac{4}{11}$. Find the probability that one at least of the events will happen.

8. Two drawings, each of two balls, are made from a bag containing 3 white and 4 black balls, the balls being replaced before the

second trial; find the chance that the first drawing will be 2 white balls and the second 2 black balls.

9. If p is the probability that a man will die within a year and q is the probability that his wife will die within a year, what are the probabilities (a) that the man will die and his wife survive, (b) that both will die, (c) that one at least will survive?

129. Mathematical Expectation. Suppose there are 100 tickets in a lottery for a prize for \$200, then since each ticket is equally likely to win the prize and the person who possessed all the tickets must win, the money value of each ticket is \$2.

If I own one ticket my chance of winning the prize is $\frac{1}{100}$. We therefore say that my mathematical expectation is $\frac{1}{100}$ of \$200 or \$2.

The mathematical expectation from any event is the product of the sum of money to be paid if the event happens and the probability that the event will happen.

Ex. 1.—I hold a ticket in a lottery in which there are 100 blanks and 20 prizes each of \$60. What is my expectation?

My chance of drawing a prize is $\frac{20}{120}$ or $\frac{1}{6}$.

$\therefore$ my expectation $= \frac{1}{6}$ of \$60 $=$ \$10.

Ex. 2.—A stake of \$24 is to be won if a player throws less than 3 in a single throw with a die. What is his expectation?

His chance of throwing less than 3 is $\frac{1}{3}$ and therefore his expectation is $\frac{1}{3}$ of \$24 or \$8.

130. The Mortality Table. The most important application of the theory of probability is to be found in the problems which are to be solved in life insurance. Primarily these problems all depend for their solution upon the duration of human life at different ages. The data which are used in the solution of these problems are given in a table which has been constructed from the observation of a great number of persons.

This table, which is called a **mortality table,** takes 100,000 persons all aged 10 years and records the number which will die each year until all are dead.

This table, which is given on page 240, shows that of these 100,000 persons of age 10, 749 will die between the ages of 10 and 11, 746 between the ages of 11 and 12, and so on.

The probability that a person will die between any two given ages can at once be found from the table.

Thus, of the 100,000 at age 10, 749 will die within one year, and therefore the probability that a person of age 10 will die within a year is $\frac{749}{100000}$ or ·00749 and the probability of living is 1 − ·00749 or ·99251.

Of the 92,637 alive at age 20 there are 85,441 alive at age 30 or 7196 have died in the interim. Therefore the probability that a person aged 20 will die within 10 years is 7196 ÷ 92,637 or ·0777, and the probability that he will live 10 years is ·9223.

131. Insurance Premiums. Suppose a man of age 25 wishes to insure his life for $1000, this sum to be paid to his heirs if he should die within a year, and we desire to find the premium which he should pay.

The mortality table shows that of 89,032 alive at age 25, 718 will die within a year.

Therefore the chance that he will die within the year $= 718 \div 89{,}032 = \cdot00806$.

Therefore the mathematical expectation of his heirs at the end of the year $= \$1000 \times \cdot00806 = \$8\cdot06$.

If money is worth 4% the present value of the expectation $= \$8\cdot06 \div 1\cdot04 = \$7\cdot75$.

Therefore the premium to be paid in advance for an insurance policy of $1000 for one year on the life of a man aged 25 is $7·75. Of course the cost of conducting the business must be added to the premium. Thus, if the premium is loaded 25% to cover expenses, the gross premium in the preceding case would be $9·69.

EXERCISE 55.

1. If 3 houses out of every 1000 of a certain class burn down every year, what would be the cost of insuring a house for \$5000 for one year if the net premium is loaded 25% to cover expenses?

2. The probability that a ship will complete a certain voyage in safety is ·998. What should be the net cost of insuring such a ship and its cargo for \$400,000?

3. If under war conditions the probability in the preceding problem is reduced to ·983, what additional net premium should be paid?

4. What is the probability that a man aged 35 will live to be 45?

5. A man aged 50 starts an enterprise requiring 10 years for its completion. What is the probability that he will see the end of the work?

6. A teacher is entitled to a pension after 40 years' service. What is the probability that a man who begins teaching at age 22 will live to take advantage of his pension?

7. A man aged 65 is to receive \$1000 at the end of a year if he is then alive. If money is worth 4%, what is the present value of his expectation?

8. A father bequeaths \$5000 to his son aged 12, the money to be paid to him when he becomes of age at 21. What is the cash value of the bequest if money is worth 5%?

9. A man is 40 years old and his son is 10 years old; what is the probability that both will live 11 years?

10. What premium per \$1000 should be paid in advance to insure a man aged 40 for one year, if money is worth 5% and if the premium is loaded 30% to cover expenses?

11. In the preceding problem what single premium should be paid to insure him for two years, if we assume that in case of death the \$1000 is paid at the end of the two years?

12. A man's age is 22 years and his wife's is 21 years when their first child is born. What is the probability that both will be alive when the child becomes of age at 21?

13. If the man and wife, in the preceding example, are both alive when the child is 10 years old, what is the probability that they will both be alive when the child is of age?

14. Upon the death of a person an insurance company agreed to pay \$1000 to his estate. If money is worth 5% and if the person's expectation of life is 20 years, what should be the annual premium payable in advance if 25% is added to cover expenses?

CHAPTER XVI

SCALES OF NOTATION

132. In the common system of notation we use nine digits and a cipher to represent any number.

The value of each digit depends upon its position with regard to the other digits.

Thus, in the number 763, the 7 represents 7 hundreds, the 6 represents 6 tens and the 3 represents 3 units or

$$763 = 7 \times 100 + 6 \times 10 + 3 = 7 \,.\, 10^2 + 6 \,.\, 10 + 3.$$

Since each digit in the number represents a multiple of some power of 10, the common system of notation is called the **denary** or **decimal** system or scale of notation. The 10 is called the **radix** of the system.

Any positive integer might be taken as the radix of a scale of notation.

Thus, if the radix is *eight* the number represented by 4716 is

$$4 \times 8^3 + 7 \times 8^2 + 1 \times 8 + 6.$$

In the common scale the greatest digit is 9, that is, one less than the radix. In the scale whose radix is *eight* the greatest digit is 7.

Similarly, in the scale of *eleven* a digit to represent " ten " units will be necessary, and for this purpose we will use the letter t, and in the scale of twelve we will use e to represent " eleven " units.

If the radix is r the number represented by

$$32015 = 3r^4 + 2r^3 + 0r^2 + 1r + 5,$$
$$= 3r^4 + 2r^3 + r + 5.$$

Similarly, if we suppose the digits of a number N, in the scale whose radix is r, to be a_0, a_1, a_2, . . . a_p, beginning with the units, then

$$N = a_p r^p + a_{p-1} r^{p-1} + \ldots + a_2 r^2 + a_1 r + a_0.$$

Here the digits a_0, a_1, a_2, . . . a_p may each have any integral value less than r, and any of them may be zero except a_p.

133. The operations of addition, subtraction, etc., are performed in any scale by methods which are similar to those used in the common scale. Thus in finding the sum of a number of addends we add the digits in each column as in the common scale and find the figure to be carried to the next column by dividing by the radix of the scale.

Thus, if we wish the sum of 6, 7 and 8 in the scale whose radix is *nine*, we add the 6, 7 and 8 which gives the number 21 of the common scale. Since 21 is equal to 2 nines and 3 units, the sum is set down as 23.

We must not read this sum as *twenty-three*, as it is not 2 tens and 3 units. The usual way is to read the digits in order, stating the radix of the scale.

Ex. **1.**—Add 4257, 3214, 5064 in the scale of *eight*.

4257
3214
5064
14557

The sum of the first column is 15 of the common scale which is equal to $1 \cdot 8 + 7$. The 7 is set down and the 1 is carried to the second column. The sum of the second column $= 13 = 1 \cdot 8 + 5$, and so on.

Ex. **2.**—Multiply 42*e*5 by 1*t*4 in the scale of *twelve*.

42*e*5
1*t*4
14*e*98
36562
42*e*5
7*t*9*te*8

The first partial product is 20 of the common scale, which is equal to $1 \cdot 12 + 8$. The second partial product is $4 \times e + 1$ or 45 of the common scale, which equals $3 \cdot 12 + 9$. The third partial product is $4 \times 2 + 3$ or 11 of the common scale, which is e of scale *twelve*, and so on.

Ex. 3.—Divide (1) $599t104$ by 7 in the scale of *eleven*.
(2) 42140 by 123 in the scale of *five*.

(1) $7)599t104$
$92t963\frac{3}{7}$.

(1) $5 \cdot 11 + 9 = 64$; $64 \div 7 = 9$ and 1 over.
$1 \cdot 11 + 9 = 20$; $20 \div 7 = 2$ and 6 over.
$6 \cdot 11 + t = 76$; $76 \div 7 = t$ and 6 over.

(2) $123)42140(243$
301
1204
1102
1020
424
41

(2) It might be thought that the first digit in the quotient is 3, but on trial we find it is only 2. Since the multiplication table is not learned in the scale of *five*, the student must not be surprised if he fails to guess the proper quotient.

Verify these two results by multiplication.

EXERCISE 56

1. Write down the first fifteen integers in the scale of *six*. In the scale of *four*. In the scale of *two*.

2. Write down all integers with two digits in the scale of *five*. In the scale of *three*.

Perform the operations indicated :

3. $1216 + 4253 + 2134 + 1032$ in the scale of 7.

4. $5678 + 4567 - 5342$ in the scale of 9.

5. $74t0 + te59 + 8e5t$ in the scale of 12.

6. $6543210 - 1234567$ in the scale of 8.

7. 1234×4321; $(516)^2$; $(24)^3$ in scale 7.

8. $45062 \div 324$ in scale 9.

9. Show that 432, 565, 708 of scale 9 are in A.P.

10. Find the average of the numbers 426, 531, 215, 316, 140 in the scale of 7.

11. Find the square root of 3521 in scale 6; of 23121 in scale 4; of $23t314$ in scale 12.

12. Find the cube of 111 in the scale of *two*.

13. Divide 17832126 by 4685 in the scale of 9 and verify the result.

14. Find the G.C.M. of 242 and 1021, both numbers being in the scale of 5.

15. Find the square of 1112, the radix being greater than 6.

16. Show that 121, 165, 264, 426 of scale 7 are in continued proportion.

134. A number which is expressed in any given scale may be transformed to any other scale.

Thus, 25 of scale *ten* $= 3 \cdot 7 + 4 = 34$ of scale *seven*.
73 of scale *ten* $= 6 \cdot 11 + 7 = 67$ of scale *eleven*.
117 of scale *ten* $= 23 \cdot 5 + 2 = (4 \cdot 5 + 3)5 + 2$
$= 4 \cdot 5^2 + 3 \cdot 5 + 2 = 432$ of scale *five*.

Ex. **1.**—Express 1235 of scale *ten* in scale *six*.

6)1235	The first division show that $1235 = 205 \times 6 + 5$,
6)205 . . . 5	the second that $205 = 34 \times 6 + 1$,
6)34 . . . 1	the third that $34 = 5 \times 6 + 4$.
5 . . . 4	$\therefore \quad 1235 = 205 \times 6 + 5$
	$= 34 \cdot 6^2 + 1 \cdot 6 + 5$
	$= 5 \cdot 6^3 + 4 \cdot 6^2 + 1 \cdot 6 + 5$
	$= 5415$ in scale *six*.

Here the given number is in the common scale and the divisions are performed in that scale. If the number is not in scale *ten*, the divisions must be performed in the scale in which the number is given

Ex. **2.**—Transform 4754 of scale *eight* to scale *twelve* and verify by transferring the result back to scale *eight*.

12)4754	8)1578
12)323 . . . 8	8)225 . . . 4
12)21 . . . 7	8)33 . . . 5
1 . . . 5	4 . . . 7

The first operation shows that 4754 of scale 8 = 1578 of scale 12. The second verifies the result as 1578 of scale 12 = 4754 of scale 8.

We see then that a number is transferred to any given scale by dividing the number and each succeeding quotient by the radix of the new scale until we obtain a quotient which is less than the radix of the new scale.

The last quotient and the remainders in order form the digits of the number in the new scale.

EXERCISE 57

Transform from the common scale the following:

1. 100 to scale 7. **2.** 237 to scale 4. **3.** 1111 to scale 12.

4. 12345 to scale 9. **5.** 99999 to scale 11. **6.** 573 to scale 2.

7. Express a million in the scale of 12.

8. Convert 4444 of scale 5 to scale 10.

9. Express 42167 in ascending powers of 9.

10. Transform 43751 from scale 12 to scale 10 and verify.

11. Express $100e$ of scale 12 in scale 9.

12. Express 231 and 452 of scale 10 in scale 7. Find their product in scale 7 and verify by reducing the result to the scale of 10.

13. Express in the scale of 7 the greatest and least numbers which can be formed with 4 digits in scale 8.

14. Reduce 3 yd. 2 ft. 4 in. to inches and reduce 324 of scale 7 to scale 10 by a similar method.

15. Find the sum of $1 + 2 \cdot 7 + 3 \cdot 7^2 + 4 \cdot 7^3 + 5 \cdot 7^4 + 6 \cdot 7^5$ by transferring the number 654321 of scale 7 to scale 10.

16. Express 567 in the scale of 5, find its square working in that scale, change the result to the scale of 10 and verify by squaring 567.

17. The sum of the three digits of a number in the scale of 7 is 11 (scale of 7), and the digit on the left exceeds the digit on the right by 2. When the number is expressed in the scale of 9 the order of the digits is reversed. Find the number in the scale of 7.

135. Radix Fractions. In the common scale the fraction ·418 denotes 4 tenths, 1 hundredth and 8 thousandths, or

$$\cdot 418 = \frac{4}{10} + \frac{1}{100} + \frac{8}{1000} = \frac{4}{10} + \frac{1}{10^2} + \frac{8}{10^3}.$$

Similarly, ·325 in scale *eight* $= \frac{3}{8} + \frac{2}{8^2} + \frac{5}{8^3}$,

and ·2133 in scale *four* $= \frac{2}{4} + \frac{1}{4^2} + \frac{3}{4^3} + \frac{3}{4^4}$,

and ·1685 in scale r $= \frac{1}{r} + \frac{6}{r^2} + \frac{8}{r^3} + \frac{5}{r^4}$

In the common scale such fractions are called **decimal fractions,** but in any other scale they are called **radix fractions.**

Ex. 1.—Reduce $\frac{1}{8}$ to a decimal fraction.

$\frac{1}{8} \times 10 = 1 + \frac{1}{4}$, | $\frac{1}{8}$ of a unit $= \frac{10}{8}$ tenths $= 1$ tenth $+ \frac{1}{4}$ of a tenth,
$\frac{1}{4} \times 10 = 2 + \frac{1}{2}$, | $\frac{1}{4}$ of a tenth $= \frac{10}{4}$ hundredths $= 2\frac{1}{2}$ hundredths,
$\frac{1}{2} \times 10 = 5$. | $\frac{1}{2}$ of a hundredth $= \frac{10}{2}$ or 5 thousandths.

$$\therefore \frac{1}{8} = \frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3} = \cdot 125.$$

Ex. 2.—Reduce $\frac{3}{8}$ to radix fractions in scale *twelve*.

$\frac{3}{8} \times 12 = 4 + \frac{1}{2}$, | $\frac{3}{8}$ of a unit $= \frac{36}{8}$ twelfths $= 4\frac{1}{2}$ twelfths,
$\frac{1}{2} \times 12 = 6$. | $\frac{1}{2}$ of a twelfth $= \frac{12}{2}$ or 6 (twelfths)^2.

$$\therefore \frac{3}{8} = \frac{4}{12} + \frac{6}{12^2} = \cdot 46 \text{ in scale } 12.$$

Ex. 3.—Transform 4275·72 from scale 9 to scale 6.

6)4275	·72
	6
6)642 . . . 2	4·32
6)106 . . . 2	6
6)15 . . . 3	4·80
2 . . . 2	6
	5·30
	6
	2·00

Here the integral part is transformed to scale 6 by successive divisions by 6 and the fractional part by successive multiplications by 6. In each case the operations are performed in scale 9. The number in the new scale is 22322·4452.

EXERCISE 58

1. Reduce $\frac{1}{4}$ to radix fractions in the scales of 6, 8, 12.

2. Reduce $\frac{3}{16}$ to a decimal fraction explaining each step in the process.

3. Transform $\frac{3}{16}$ from the common scale to radix fractions in the scales of 4, 6, 8 and 12.

4. Express $\frac{23}{248}$ of scale 10 by a series of radix fractions in the scale of 8.

5. The fractions $\frac{5}{12}$, $\frac{3}{23}$, $\frac{43}{100}$ are in the scale of *six*. Change them into decimal fractions.

6. Transform ·21875 from the common scale to the scale of 8 and to the scale of 6.

7. Transform 333·333 from scale 4 to scale 8.

8. Multiply 35·3 of scale 6 by 35·3 of scale 12, giving the answer in scale 10.

9. In what scales can $\frac{7}{24}$ of scale 10 be represented by finite radix fractions?

10. Divide 252710 by 249 in scale 12. Test the result by reducing dividend, divisor and quotient to scale 10.

11. Express ·5 in scale 7 and verify the result by summing an infinite G.P.

12. Find the first four radix fractions when $\frac{5}{15}$ of the common scale is reduced to scale 7.

136. Theorems on Divisibility. It is sometimes possible to determine, without actual division, when a number is divisible by certain simple numbers. It is assumed that the number is in the common scale, unless otherwise stated.

Let the digits of a number N be $a, b, c, \ldots$ beginning with the units.

Then $$N = a + 10b + 10^2c + 10^3d + \ldots$$

If it is required to know under what condition this number is divisible by 4, we have

$$\frac{N}{4} = \frac{a+10b+10^2c+10^3d+\ldots}{4} = \frac{a+10b}{4} + \frac{10^2c+10^3d+\ldots}{4}.$$

Since 10^2, 10^3, 10^4, . . . are always divisible by 4, the last fraction is an integer for all values of c, d, e . . .

$\therefore$ if $a + 10b$ is divisible by 4, N is also divisible.

That is, any integer is divisible by 4 if the number formed from the last two digits is divisible by 4.

Thus, 428756 is divisible by 4 since 56 is divisible.

Under what conditions is a number divisible by 2, 5, 8, 10, 16?

137. Divisibility by 9. If S is the sum of the digits of the given number N, then

$$N = a + 10b + 10^2c + 10^3d + \ldots$$
$$S = a + b + c + d + \ldots$$
$$\therefore \quad N - S = 9b + 99c + 999d + \ldots$$

The expression on the right is divisible by 9 for all values of b, c, d . . . , therefore $N - S$ is divisible by 9.

It therefore follows that

(1) The difference between any number and the sum of its digits is divisible by 9.

(2) Any number is divisible by 9 if the sum of its digits is divisible by 9.

(3) Since $\frac{N - S}{9}$ is an integer, the remainder on dividing any number by 9 is the same as the remainder on dividing the sum of its digits by 9.

Thus, in the number 6537135, $s = 30$. The remainder on dividing 30 by 9 is 3, therefore when 6537135 is divided by 9 the remainder is also 3.

It is not necessary to find the actual sum of the digits as we ar interested only in the remainder when the sum is divided by 9.

In adding, therefore, the 9's may be discarded as we reach them. Thus $6 + 5 = 11$, which we may call 2 when the 9 is discarded; then $2 + 3 + 7 = 12$, which we may call 3, and so on.

This process of finding the remainder when a number is divided by 9 is called "**casting out the nines.**"

138. Testing the Accuracy of Arithmetical Operations. The rule for "casting out the nines" is frequently used to test the accuracy of results in addition and multiplication.

Let N_1 and N_2 be any two integers and let the remainders on dividing them by 9 be r_1 and r_2 respectively.

$\therefore$ $N_1 = 9a + r_1$ and $N_2 = 9b + r_2$, where a and b are integers.

$$\therefore \quad N_1 + N_2 = 9(a + b) + r_1 + r_2.$$

$\therefore$ the remainder on dividing $N_1 + N_2$ by 9 is the same as on dividing $r_1 + r_2$ by 9, as $9(a + b)$ is a multiple of 9.

Also $$N_1 N_2 = 81ab + 9(ar_2 + br_1) + r_1 r_2.$$

$\therefore$ the remainder on dividing the product of the numbers by 9 is the same as on dividing $r_1 r_2$ by 9.

Ex.—Test the accuracy of the following by "casting out the nines," (a) being addition and (b) multiplication.

(a)		(b)	
428	(5)	6897	(3)
586	(1)	89	(8)
762	(6)		
		62173	
1776	(3)	55176	
		613933	(7)

The remainder in each case on "casting out the nines" is given in brackets. In (a) the remainder on dividing $5 + 1 + 6$ by 9 is 3 and the addition is *likely* correct. In (b) the remainder on dividing 3×8 by 9 is 6 and the multiplication is incorrect.

139. Divisibility by $r - 1$ in the Scale of r. The results of Art. 137 may be generalized to apply to any scale.

Let $$N = a + br + cr^2 + dr^3 + \ldots$$

and $$S = a + b + c + d + \ldots$$

$$\therefore \quad N - S = b(r - 1) + c(r^2 - 1) + d(r^3 - 1) + \ldots$$

The expression on the right is always divisible by $r - 1$,

$$\therefore \quad N - S \text{ is divisible by } r - 1.$$

$\therefore$ the difference of any number and the sum of its digits is divisible by $r - 1$ in the scale of r.

Also the number is divisible by $r - 1$ if the sum of its digits is divisible by $r - 1$.

The remainder on dividing any number by $r - 1$ is the same as on dividing the sum of its digits by $r - 1$.

140. Divisibility by $r + 1$ in the scale of r. Any number in the scale whose radix is r is divisible by $r + 1$ if the difference of the sums of the digits in the odd and even places is divisible by $r + 1$.

Let $N = a + br + cr^2 + dr^3 + \ldots$

Let $D = a - b + c - d + \ldots$

$\therefore N - D = b(r + 1) + c(r^2 - 1) + d(r^3 + 1) + \ldots$

But $r + 1$, $r^2 - 1$, $r^3 + 1$, . . . are all divisible by $r + 1$,

$\therefore N - D$ is divisible by $r + 1$.

$\therefore N$ is divisible by $r + 1$ if D is divisible by $r + 1$.

In the particular case when $r = 10$, this theorem gives the condition that a number is divisible by 11 in the common scale.

141. Miscellaneous examples.

Ex. **1.**—Show that 12544 is a perfect square in any scale whose radix is greater than 5

Let r be the radix, then

$$12544 = r^4 + 2r^3 + 5r^2 + 4r + 4$$
$$= (r^2 + r + 2)^2 = (112)^2.$$

Therefore the number is the square of 112. Of course the radix must be greater than 5 as that digit appears in the number.

Solve also by finding the square root of 12544 arithmetically.

Ex. **2.**—In what scale is 446 (scale 10) expressed by 676 ?

Let r be the radix of the required scale.
Then $446 = 6r^2 + 7r + 6$. Solving we find $r = 8$.

$\therefore$ the radix of the scale is 8.

Or, since 676 is greater than 446 if both were in the same scale, we see that the required radix is less than 10. But a digit 7 appears in the new number, therefore the radix is greater than 7. Therefore the radix is either 8 or 9.

Now find by trial whether it is 8 or 9.

Ex. 3.—Which of the weights 1 lb., 2 lb., 2^2 lb., 2^3 lb., . . . must be taken to weigh 37 lb. ?

When 37 is expressed in the scale whose radix is 2 we get 100101.

$\therefore \quad 37 = 1\cdot2^5 + 1\cdot2^2 + 1.$

$\therefore$ the weights required are 1 lb., 2^2 lb., 2^5 lb.

EXERCISE 59

1. What is the remainder when each of the following numbers is divided by 9 : 62879, 49832, 57904317 ?

2. If the number $624a596$ of the common scale is divisible by 9, what is the value of a ?

3. Which of the following are incorrect :

(1) $586 \times 328 = 192208$; (2) $444^2 = 187136$;
(3) $79^3 = 492939$; (4) $13208148 \div 386 = 34218$?

4. In the scale of 8 show that any integer is divisible by 4 if its units digit is 4 or 0.

5. In the scale of 12 what is the condition that a number is divisible by 3 ? 4 ? 6 ? 8 ?

6. Show that any number of six digits in the common scale, whose first and fourth, second and fifth, third and sixth digits are alike is divisible by 7, 11 and 13.

7. In what scale is 95 of scale 10 denoted by 137 ?

8. In what scale is 529 of scale 10 denoted by 20101 ?

9. In what scale is the square of 34 equal to 1420 of the same scale ? In what scale is 311 the square of 14 ?

10. Show that 173, 352, 531 are in A.P. in any scale whose radix is greater than 7.

11. Show that 1·44 and 12343·21 are perfect squares in any scale whose radix is greater than 4.

12. Find a number of two digits in the scale of 5 which is doubled by reversing its digits.

13. What decimal number of two digits is expressed by the same two digits in the scales of 5 and 7?

14. In what scale is the decimal ·75 represented by ·43?

15. If ·16, ·2, ·28 are in G.P., find the radix.

16. If 332, 461, 600 are in A.P., find the radix.

17. If the radix is greater than three, show that 1331 is always a perfect cube.

18. Of the weights 1, 2, 2^2, 2^3, . . . lb., which must be taken to weigh half of a ton, not more than one of each being chosen?

19. Express $\cdot\dot{5}$ of scale 8 as an infinite G.P. and thus find its value as a common fraction.

20. Prove that the difference of any two numbers made up of the same digits but in a different order is divisible by 9 in the common scale.

21. The sides of a rectangle are 14 ft. 5 in. and 10 ft. 2 in. Find the area by multiplication in the scale of 12.

22. If a number in the common scale contains n digits, show that its square contains $2n$ or $2n - 1$ digits.

23. Multiply ·2343 by 332, both being in scale 6.

24. If the sum of the digits in the even places of a number equals the sum of the digits in the odd places, prove that the number is divisible by $r + 1$, r being the radix of the scale.

25. What errors will the test by "casting out the nines" fail to detect in a problem in multiplication?

26. Find, by multiplication in the scale of 12, the volume of a rectangular solid whose dimensions are 3 ft. 6 in., 4 ft. 3 in., 6 ft. 8 in.; expressing the result as a whole number and radix fraction in the same scale.

MISCELLANEOUS EXAMPLES

1. If a, b, c, d be in H.P., show that $3(b-a)(d-c) = (c-b)(d-a)$.

2. If the roots of $(b-c)x^2 + (c-a)x + a - b = 0$ are equal, show that a, b, c are in A.P.

3. Assuming that x and y are unequal, solve

$$x - y = 1 - z,$$
$$3(x^2 - y^2) = 5(1 - z^2),$$
$$7(x^3 - y^3) = 19(1 - z^3).$$

4. Solve $x^2 - 4y^2 + x + 3y = 2x - y = 1$.

5. If $x^2 - yz = a$, $y^2 - zx = b$, $z^2 - xy = c$, show that

$$\frac{x}{a^2 - bc} = \frac{y}{b^2 - ca} = \frac{z}{c^2 - ab}.$$

6. Solve $\quad x + \frac{1}{y} = \frac{3}{2}, \; y + \frac{1}{z} = \frac{7}{3}, \; z + \frac{1}{x} = 4.$

7. Construct the cubic function of x which equals 1 when $x = 1$, equals -5 when $x = -1$, equals 13 when $x = 2$ and equals -11 when $x = -2$.

8. If $\frac{y}{x} + \frac{x}{z} = a$, $\frac{z}{y} + \frac{y}{x} = b$, $\frac{x}{z} + \frac{z}{y} = c$, show that

$$(a + b - c)(b + c - a)(c + a - b) = 8.$$

9. Find the limiting value of $\frac{x^2 - 7x + 10}{x^2 - 9x + 14}$, when $x = 2$.

10. Solve $\quad \frac{4^x}{2^{x+y}} = 8, \; \frac{9^{x+y}}{3^{5y}} = 243.$

11. Factor $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$.

12. If $x(b-c)+y(c-a)+z(a-b)=0$, show that

$$(a)\quad \frac{bz-cy}{b-c}=\frac{cx-az}{c-a}=\frac{ay-bx}{a-b}.$$

$$(b)\quad \frac{y-z}{b-c}=\frac{z-x}{c-a}=\frac{x-y}{a-b}.$$

13. In an A.P., if p times the pth term equals q times the qth term show that the $(p+q)$th term equals 0.

14. If $a+\frac{1}{a}=l$, $b+\frac{1}{b}=m$, $c+\frac{1}{c}=n$, $abc=1$, show that

$$l^2+m^2+n^2=lmn+4.$$

15. In how many ways can a foot-rule be divided into three pieces so that each piece contains an exact number of inches?

16. Solve $(x+9)(x-3)(x-7)(x+5)=385$.

17. If $\frac{p}{bc-a^2}=\frac{q}{ca-b^2}=\frac{r}{ab-c^2}$, then $\frac{a}{qr-p^2}=\frac{b}{rp-q^2}=\frac{c}{pq-r^2}$.

18. If x^3+2ax^2+bx+c is divisible by x^2+ax+d, find b and c in terms of a and d.

19. The two middle terms of an A.P. of $2n$ terms are a and b. Find the difference between the sum of the last n terms and that of the first n terms.

20. If a, x, y, b are in A.P and c^3, x, y, d^3 are in G.P., show that $a+b=cd(c+d)$.

21. If $y=x+x^2+x^3+\ldots$ to inf., prove that $x=y-y^2+y^3-y^4+\ldots$ to inf., x and y being less than 1.

22. A rod whose length is 5 ft. is broken into three parts, out of which a right-angled triangle can be formed whose area is 120 sq. in. Find the lengths of the parts.

23. If x is real, show that $\frac{2x^2+6x+3}{2x+1}$ must be greater than 3 or less than 1.

24. The number of shot in the base of a rectangular pile is 800, and the sixth course from the base contains 495; find the number in the complete pile.

25. Find the least positive value of $4x + \frac{9}{x}$, for real values of x.

26. If $x + 2\left(x - \frac{1}{n}\right) + 3\left(x - \frac{2}{n}\right) + \ldots$ to $n + 1$ terms $= 0$, find x.

27. Find the coefficient of x^5 in the product

$$(1 - x)\left(1 + nx + \frac{n(n+1)}{1.2}x^2 + \frac{n(n+1)(n+2)}{1.2.3}x^3 + \ldots\right).$$

28. If $a + b + c = 0$, show that $2(a^5 + b^5 + c^5) = 5abc(a^2 + b^2 + c^2)$.

29. If a, b, c are in H.P., then

$a(b - c)x^2 + b(c - a)xy + c(a - b)y^2$ is a perfect square.

30. Prove that ${}_{2n+4}C_{n+2} + {}_{2n+4}C_{n+3} = {}_{2n+5}C_{n+3}$.

31. If $a + b + c = 0$, prove that $\frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab} = 1$.

32. Solve $\sqrt{3x - 24} + \sqrt{x + 7} = \sqrt{3x - 14} + \sqrt{x - 3}$.

33. The product of two quantities varies as the excess of their sum over 3, and when one of the quantities is equal to 2 so is the other. Find what other value they may have simultaneously.

34. The equation $\frac{8x + 1}{4x^3 - x} = \frac{A}{2x - 1} + \frac{B}{x} + \frac{C}{2x + 1}$ is an identity for what values of A, B, C?

35. When h is infinitely small, find the limiting value of

$$\frac{(x + h)^n - x^n}{h}.$$

36. If b is half the harmonic mean between a and c, prove that

$$a^3 - b^3 + c^3 + 3abc = (a - b + c)^3.$$

37. Solve $x^2 - yz = 64$, $y^2 - zx = 88$, $z^2 - xy = 4$.

38. Eliminate x from $ax^3 + bx^2 + c = 0$, $a_1x^3 + b_1x^2 + c_1 = 0$.

39. If $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} = 0$, prove $(a + b + c)^3 = 27abc$.

40. If $\frac{a}{p(px - qy - rz)} = \frac{b}{q(qy - rz - px)} = \frac{c}{r(rz - px - qy)}$,

prove $\frac{x}{br + cq} = \frac{y}{cp + ar} = \frac{z}{aq + bp}$,

and $\frac{p}{a(ax - by - cz)} = \frac{q}{b(by - cz - ax)} = \frac{r}{c(cz - ax - by)}$.

41. Eliminate x, y, z from $x + y + z = a$, $x^2 + y^2 + z^2 = b^2$, $x^3 + y^3 + z^3 - 3xyz = c^3$.

42. Solve $(a - x)^3 + (b - x)^3 = (a + b - 2x)^3$.

43. If $a^3 + b^3 + c^3 = 3abc$, prove that $a + b + c = 0$ or $a = b = c$.

44. If $\left(a + \frac{1}{a}\right)^2 = 3$, prove that $a^3 + \frac{1}{a^3} = 0$.

45. The volume of a rectangular block is 6 cu. ft., the total surface is 22 sq. ft. and the total length of the edges is 24 ft. Find the dimensions.

46. Solve $2(x-1)+3(y-2)+(z-3)=3(x-1)+y-2+2(z-3)=0$, and $x^2 + y^2 + z^2 = 14$.

47. Find the two roots of $2x^2 - 1{\cdot}732x - 2{\cdot}75 = 0$, each correct to the second decimal place.

48. If $S_1 = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots$ to infinity,

and $S_2 = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \ldots$ to infinity,

show that $S_1 : S_2 = 2^{n-1} : 2^{n-1} - 1$.

49 Solve $x - \sqrt{yz} = 38$, $y - \sqrt{zx} = -22$, $z - \sqrt{xy} = -2$.

50. Eliminate x and y from the equations

$$x + y = a,\ x^3 + y^3 = b^3,\ x^4 + y^4 = c^4.$$

51. Express $3x^3 + 7x^2 - 15x - 31$ as a sum of powers of $x - 3$.

52. If the roots of $x^2 + ax + b = 0$ are proportional to the squares of the roots of $x^2 + 4x - 6 = 0$, prove that $9a^2 = 196b$.

53. If the pth and qth terms of an A.P. are q^2 and p^2 respectively, find the sum of $(2p + 2q - 1)$ terms.

54. A loan of \$10,000 is to be repaid both as to principal and interest by the payment of a certain sum at the end of each year for the first five years and double that sum at the end of each year for the next five years. If the interest rate is 5%, find the sum.

55. Show that $x^2 + px + q = 0$ and $px^2 + qx + 1 = 0$, have a common root if $p^2 + q^2 + 1 = pq + p + q$.

56. Find three quadratic factors of

$$a^3b^3 + b^3c^3 + c^3a^3 + a^4bc + b^4ca + c^4ab + 2a^2b^2c^2.$$

57. Solve $x^3 + y^3 + z^3 = 73$, $x + y + z = 7$, $xyz = 8$.

58. Sum to n terms $1 + 11 + 111 + 1111 + \ldots$

59. If $\dfrac{a^{\frac{1}{n}} - mb^{\frac{1}{n}}}{a^{\frac{1}{n}} + mb^{\frac{1}{n}}} = \dfrac{a^{\frac{1}{2n}} - m(a-b)^{\frac{1}{2n}}}{a^{\frac{1}{2n}} + m(a-b)^{\frac{1}{2n}}}$, find $a : b$.

60. Sum to n terms $(a+b)+(a^2+ab+b^2)+(a^3+a^2b+ab^2+b^3)+\ldots$

61. Find the number of factors of 17640.

62. Find the condition that the roots of $ax^2 + 2bx + c = 0$ may be formed from the roots of $px^2 + 2qx + r = 0$ by adding the same quantity to each root.

63. If $a + b + c = 0$, prove that

(i) $\dfrac{a^5 + b^5 + c^5}{5} = \dfrac{a^3 + b^3 + c^3}{3} \cdot \dfrac{a^2 + b^2 + c^2}{2}$.

(ii) $\dfrac{a^7 + b^7 + c^7}{7} = \dfrac{a^5 + b^5 + c^5}{5} \cdot \dfrac{a^2 + b^2 + c^2}{2}$.

64. If the sums of the squares of the roots of $x^2 + (m + 1)x = 3$, $2x^2 + 2(m + 2)x = 3$, $x^2 + (m + 3)x = 1$ are in G.P., find m.

65. Find the sum of n terms of $1 + 2^2 + 3 + 4^2 + 5 + 6^2 + \ldots$, (i) when n is even, (ii) when n is odd.

66. A G.P. consists of $4n$ terms. Show that the ratio of the sum of the first n terms and the last n terms is to the remaining $2n$ terms as $r^{2n} - r^n + 1$ to r^n.

67. Find the condition that the two values of $y : z$ obtained from $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = lx + my + nz = 0$ may be equal.

68. The common difference of an A.P. is d and the sum of n terms is S; show that the sum of their squares is $\dfrac{S^2}{n} + \dfrac{n}{12}(n^2 - 1)d^2$.

69. A besieged fortress is held by 5700 men, who have provisions for 66 days. If the garrison loses 20 men each day, for how many days can the provisions hold out?

70. Find the limiting value of $\dfrac{2x^3 - 5x^2 - 4x + 12}{x^3 - 12x + 16}$ when $x = 2$.

71. Determine a, b, c, d, so that the nth term in the expansion of $(a + bx + cx^2 + dx^3)(1 - x)^{-4}$ may be n^3x^{n-1}.

72. If the sum of m terms of an A.P. is to the sum of n terms as $m^2 : n^2$, show that the mth term is to the nth term as $2m - 1$ to $2n - 1$.

73. Show that

$$\frac{1}{2}\sqrt[3]{5} = 1 - \frac{1}{5} + \frac{1.4}{5.10} - \frac{1.4.7}{5.10.15} + \frac{1.4.7.10}{5.10.15.20} - \ldots$$

74. If $f(x) = \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$, prove that $f(x + y) = \dfrac{f(x) + f(y)}{1 + f(x)f(y)}$.

75. Solve $xy = 2x + y - 1$, $yz = 3y + 2z - 8$, $zx = 4z + 3x - 8$.

76. If $mx_1^2 + ny_1^2 = mx_2^2 + ny_2^2 = a^2$ and $mx_1x_2 + ny_1y_2 = 0$, show that $x_1^2 + x_2^2 = \dfrac{a^2}{m}$ and $y_1^2 + y_2^2 = \dfrac{a^2}{n}$.

77. Find the number of combinations, taken n at a time, of $2n$ things of which n are alike.

78. Find the number of the common scale which has the same two digits when expressed in the scales of 7 and 9.

79. Show that if $ax^3 + bx^2 + cx + d$ is a perfect cube, then $b^2 = 3ac$ and $c^2 = 3bd$.

80. If $a^2 + b^2 + c^2 = a_1^2 + b_1^2 + c_1^2 = 1$, all the letters representing real quantities, show that $aa_1 + bb_1 + cc_1 < 1$.

81. Solve $2x^3 - 15x^2 + 37x - 30 = 0$, being given that the roots are in A.P.

82. Show that $\dfrac{5}{6} + \dfrac{5.7}{6.9} + \dfrac{5.7.9}{6.9.12} + \ldots = 3\sqrt{3} - 2$.

83. Express $\dfrac{x}{1 - 5x + 6x^2}$ as the difference of two fractions and thus find the coefficient of x^r when $x(1 - 5x + 6x^2)^{-1}$ is expanded in ascending powers of x.

84. Show that if a and b be such that the sum of the squares of the three arithmetic means inserted between a and b is $a^2 + b^2$, then the sum of the cubes of these means is $\frac{1}{2}(a + b)^3$.

85. If $(1 - x)^{-m} = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots$, prove that
$C_0 + C_1 + C_2 + \ldots + C_k = \lfloor\underline{m + k} \div \lfloor\underline{m} \lfloor\underline{k}$.

86. Show that $\dfrac{1}{4}\sqrt{17} = 1 + \dfrac{1}{2} \cdot \dfrac{1}{2^4} - \dfrac{1}{2.4} \cdot \dfrac{1}{2^8} + \dfrac{1.3}{2.4.6} \cdot \dfrac{1}{2^{12}} - \ldots$

87. Find the number of combinations of 4 letters 5 at a time when each letter may be repeated any number of times.

88. Find the coefficient of x^5 in $(1 + x + x^2 + x^3 + x^4)^6$ and in $(1 + x + x^2 + x^3 + x^4 + x^5)^8$.

89. Sum $1n(n+1) + 2(n-1)n + 3(n-2)(n-1) + \ldots + n \,.\, 1 \,.\, 2$.

90. If n is a positive integer, show by mathematical induction that $2 \,.\, 7^n + 3 \,.\, 5^n - 5$ is divisible by 24.

91. If $a_1, a_2, a_3, \ldots a_n$ be in H.P. prove that

$$a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_{n-1}a_n = (n-1)a_1a_n.$$

92. If $x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}$, then $x^2y^2z^2 = 1$ or $x = y = z$.

93. Find a scale of notation in which the number denoted by 1155 is divisible by that denoted by 12.

94. The rates of two trains are in the ratio $a : b$. If they start at the same time from opposite ends of the same road, show that the ratio of the times they take after meeting to finish the journey is $b^2 : a^2$.

95. If $\frac{x}{a(y+z)} = \frac{y}{b(z+x)} = \frac{z}{c(x+y)}$, then $\frac{x^2-y^2}{a-b} = \frac{y^2-z^2}{b-c} = \frac{z^2-x^2}{c-a}$.

96. Find an A.P., whose first term is 1, in which the sum of the first half of any even number of terms bears a constant ratio to the sum of the second half.

97. If S_n be the sum of n terms of $a + ar + ar^2 \ldots$, show that the sum of the products of every pair of terms is $\frac{r}{r+1} \cdot S_n \,.\, S_{n-1}$.

98. In the scale whose radix is s show that the square of $rrrr$ is $rrrq0001$, where q, r, s are consecutive integers.

99. Prove that $2^n = 1 + \frac{(n+1)n}{1 \,.\, 2} + \frac{(n+1)n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \ldots$

100. If $x^4 + y^4 + z^4 + x^2y^2 + y^2z^2 + z^2x^2 = 2xyz(x + y + z)$ and x, y, z are real quantities, show that $x = y = z$.

101. Eliminate m and n from the equations

$$y = mx + \frac{a}{m}, \; y = nx + \frac{a}{n}, \; m - n = 1 + mn.$$

102. Sum $1 + \frac{7}{24} + \frac{7 \cdot 28}{24 \cdot 48} + \frac{7 \cdot 28 \cdot 49}{24 \cdot 48 \cdot 72} + \ldots$ to infinity.

103. Show that $16^n + 10n - 1$ is divisible by 25, if n is a positive integer.

104. Find the condition that one root of $x^3 + px + q = 0$ may be the reciprocal of one root of $1 + p_1x^2 + q_1x^3 = 0$.

105. What should an investor pay for a \$100 Dominion of Canada government bond, bearing interest at 5% payable yearly and redeemable in 20 years, if he wishes to make 5·4% per annum on his investment?

106. The number of vibrations made by a wire in a unit of time varies (a) inversely as the length of the wire, (b) inversely as the diameter of the wire, (c) inversely as the square root of the density of the wire, and (d) *directly* as the square root of the weight with which the wire is stretched. If a wire 3 metres long, 1·5 mm. in diameter, with a density of 2·25, and stretched by a weight of 81 kilograms, vibrates 288 times per second, how many vibrations will a wire 7·2 meters in length, 2·5 mm. in diameter, with a density of 4·84 and stretched by a weight of 121 kilograms make in 1 second?

107. If $5 + \sqrt{2}$ is a root of $3x^4 - 25x^3 + 7x^2 + 235x - 276 = 0$, find the other roots.

108. If $(b-x)(c-y) = a^2$, $(c-x)(a-y) = b^2$, $(a-x)(b-y) = c^2$, show that $x = y = a + b + c$ and that $ab + bc + ca = 0$.

109. Find the number of (i) selections, (ii) arrangements of the letters of the word *blossoms* taken 4 at a time.

110. Solve $\sqrt{4x + \sqrt{7x + 2}} - 2\sqrt{x} = 1$.

111. If the sum of n terms of a series is $2n^3 - 8n^2 - n$, find the nth term.

112. One root of $(x - a + b)(x - b + c) = (d - a + b)(d - b + c)$ is evidently $x = d$; find the other root.

113. Solve $(x^2 + y^2)(x^3 - y^3) = 7955$, $x - y = 5$.

114. Show that the coefficient of x^{2n+1} in the expansion of

$$(1 + x + x^2 + \ldots + x^{n-1})^3 \text{ is } \tfrac{1}{2}(n - 2)(n - 3).$$

115. Find the coefficient of x^{12} in $(1 - x + x^2 - x^3)^{-5}$.

116. If ${}_nC_{r+1} = \frac{5}{24}\,{}_{n+1}C_r = \frac{11}{8}\,{}_{n-1}C_r$, find n and r.

117. In how many ways can a selection of 7 things be made from 12 things, of which 2 are alike of one kind, 3 alike of another and the rest all different?

118. Show that the coefficient of x^{n+r-1} in the expansion of $\frac{(1+x)^n}{(1-x)^2}$ is $2^{n-1}(n+2r)$.

119. Find the sum of the first n coefficients in the expansion of $(1+x)^n(1-x)^{-2}$.

120. If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$, prove that

$$\frac{1}{a^{2n+1}}+\frac{1}{b^{2n+1}}+\frac{1}{c^{2n+1}}=\frac{1}{(a+b+c)^{2n+1}}.$$

121. If a, x, y are each less than 1, show that

$$\frac{1}{1-x}+\frac{y}{1-ax}+\frac{y^2}{1-a^2x}+\ldots=\frac{1}{1-y}+\frac{x}{1-ay}+\frac{x^2}{1-a^2y}+\ldots$$

122. Show that the product of any number of factors, each of which is the sum of two squares, may be expressed as the sum of two squares.

123. Find the digits a and b in order that the number $13ab456$ may be divisible by 792.

124. Find the sum of n terms of the series formed by taking the 1st, 2nd, 4th, 8th, 16th . . . terms of the A.P. $a+(a+d)+(a+2d)+\ldots$

125. Show that, by giving an appropriate real value to x, that $\frac{4x^2+36x+9}{12x^2+8x+1}$ can be made to assume any real value.

126. If x is numerically less than 1, show that

$$\frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^3}{1-x^3}+\ldots=\frac{x}{(1-x)^2}+\frac{x^2}{(1-x^2)^2}+\frac{x^3}{(1-x^3)^2}+\ldots$$

127. If $x^2(y+z)=a^2$, $y^2(z+x)=b^2$, $z^2(x+y)=c^2$, $xyz=abc$, show that $a^2+b^2+c^2+2abc=1$.

128. If a, b, c, d are all real and positive and if

$$a^4+b^4+c^4+d^4=4abcd, \text{ then } a=b=c=d.$$

129. If the receipts on a railway vary as the excess of the speed above 20 miles per hour, while the expenses vary as the square of that excess, find the speed at which the profit will be greatest if at 40 miles per hour the expenses are just covered.

130. Show that ${}_{4n}C_{2n} : {}_{2n}C_n = \dfrac{1 \cdot 3 \cdot 5 \ldots (4n-1)}{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\}^2}$.

131. If b is so small compared with a that $b^2, b^3 \ldots$ may be neglected, find the sum of n terms of the H.P. $\dfrac{1}{a+b} + \dfrac{1}{a+2b} + \ldots$

132. If S_a denote the sum of n terms of the natural numbers beginning with a, find $S_1 + S_2 + S_3 + \ldots + S_m$.

133. Show that, if $a + b + c = 0$, then

$$\left(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right)\left(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}\right) = 9.$$

134. A man writes on an examination in which there are four papers with a maximum of m marks for each. In how many ways can he get half marks on the examination?

135. Show that

$$\frac{n^2}{1^2} - \frac{n^2(n^2-1^2)}{1^2 \cdot 2^2} + \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2 \cdot 2^2 \cdot 3^2} - \ldots$$

to n terms is unity, n being a positive integer.

136. The $(l+1)$th, $(m+1)$th, $(n+1)$th terms of an A.P. are respectively equal to the $(r+1)$th, $(s+1)$th, $(t+1)$th terms of an H.P. and the first term of each is 1. Prove that

$$rsn(l-m) + stl(m-n) + trm(n-l) = 0.$$

137. Prove that the sum of the first r coefficients in the expansion of $(1-x)^{\frac{m}{n}}$ is equal to the rth coefficient in the expansion of $(1-x)^{\frac{m-n}{n}}$.

138. Show that any vulgar fraction will reduce to a terminating radix fraction provided the denominator contains no factor which is not a factor of the radix.

139. Sum $(n+1)(n+2)(n+3) \cdot 1 \cdot 2 + n(n+1)(n+2) \cdot 2 \cdot 3 + \ldots + 1 \cdot 2 \cdot 3(n+1)(n+2)$.

140. If m like cards are to be placed in $n+1$ numbered envelopes and no restriction is to be made as to the number to be placed in any envelope, find the number of ways in which the distribution can be made.

141. If $b(a^2 + c^2 + 6ac) = 2(ab^2 + b^2c + c^2a + a^2c)$, show that a, b, c are either in A.P. or in H.P.

142. A Norman window consists of a rectangle surmounted by a semicircle. Find the width of the window having perimeter p which will admit the greatest amount of light.

143. Sum $1^2 - 3^2 + 5^2 - 7^2 + \ldots$ to n terms, (i) when n is even, (ii) when n is odd. Show that in either case the sum is

$$\tfrac{1}{2}\{(4n^2 - 1)(-1)^{n+1} - 1\}.$$

144. Show that $2n$ persons may be seated at two round tables, n persons at each in $\dfrac{\lfloor 2n}{n^2}$ ways.

145. If a, b, c are such that the equations

$$ax + by + c = 0,\ bx + cy + a = 0,\ cx + ay + b = 0,$$

are consistent and every pair of them are independent as regards x and y, prove that $x^3 = y^3 = 1$ and $xy = 1$.

146. If S_1, S_2, S_3 are the sums of n terms of three arithmetic series of which the first terms are unity and the common differences are in H.P., then $n = \dfrac{2S_1S_3 - S_1S_2 - S_2S_3}{S_1 + S_3 - 2S_2}$.

147. If a, b, c are the pth, qth, rth terms respectively of both a G.P. and an H.P., show that

$$a(b - c)\log a + b(c - a)\log b + c(a - b)\log c = 0.$$

148. If $x + y + z = xyz$, show that the sum and product of

$$\frac{2x}{1 - x^2},\ \frac{2y}{1 - y^2},\ \frac{2z}{1 - z^2} \text{ are equal.}$$

149. If $f(x) = \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$ and $F(x) = \dfrac{2}{a^x + a^{-x}}$,

show that $F(x + y) = \dfrac{F(x)F(y)}{1 + f(x)f(y)}$.

150. An A.P., a G.P. and an H.P. have each a and b for their first two terms. Show that the $(n + 2)$th terms will be in G.P. if

$$\frac{b^{2n+2} - a^{2n+2}}{ab(b^{2n} - a^{2n})} = \frac{n + 1}{n}.$$

151. Find the sum to n terms of the series

$$\frac{4}{1 \cdot 5} + \frac{9}{5 \cdot 14} + \frac{16}{14 \cdot 30} + \frac{25}{30 \cdot 55} + \frac{36}{55 \cdot 91} + \ldots$$

152. If $(x + 1)^2$ is greater than $5x - 1$ and less than $7x - 3$, find the integral value of x.

153. If $p_r = \dfrac{1 \cdot 3 \cdot 5 \ldots (2r - 1)}{2^r \lfloor r}$, show that

(i) $p_{2n+1} + p_1 p_{2n} + p_2 p_{2n-1} + \ldots + p_n p_{n+1} = \frac{1}{2}$.

(ii) $2\{p_{2n} - p_1 p_{2n-1} + p_2 p_{2n-2} - \ldots + (-1)^{n-1} p_{n-1} p_{n+1}\}$
$= p_n + (-1)^{n-1} p_n^2$.

154. If $\dfrac{1}{a} + \dfrac{1}{c} + \dfrac{1}{a - b} + \dfrac{1}{c - b} = 0$, prove that a, b, c are in harmonic progression, or $b = a + c$.

155. If the roots of $x^3 + ax^2 + bx + c = 0$ are in A.P., prove that

$$2a^3 - 9ab + 27c = 0.$$

156. If $\dfrac{a^2}{l} + \dfrac{b^2}{m} + \dfrac{c^2}{n} = 0$, show that the equations $lx^2 + my^2 + nz^2 = 0$, $ax + by + cz = 0$ will have two equal sets of values for the ratios $x : y : z$, and then $\dfrac{lx}{a} = \dfrac{my}{b} = \dfrac{nz}{c}$.

157. Eliminate x, y, z from $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = \dfrac{1}{a}$, $x + y + z = b$, $x^2 + y^2 + z^2 = c^2$, $x^3 + y^3 + z^3 = d^3$.

158. Sum to n terms

$$1 + 2(2 + 3) + 3(4 + 5 + 6) + 4(7 + 8 + 9 + 10) + \ldots$$

159. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, show that

(i) $C_1 + 2C_2 x + 3C_3 x^2 + \ldots + nC_n x^{n-1} = n(1 + x)^{n-1}$.

(ii) $C_1^2 + 2C_2^2 + 3C_3^2 + \ldots + nC_n^2 = \dfrac{\lfloor 2n - 1}{\lfloor n - 1 \; \lfloor n - 1}$.

160. Show that $\dfrac{C_1^2 + 2C_2^2 + 3C_3^2 + \ldots nC_n^2}{C_0^2 + C_1^2 + C_2^2 + \ldots C_n^2} = \dfrac{n}{2}$.

161. Solve $12x^5 - 8x^4 - 45x^3 + 45x^2 + 8x - 12 = 0$.

162. If a, b, are the roots of $x^2 + px + 1 = 0$, and c, d are the roots of $x^2 + qx + 1 = 0$, show that

$$(a - c)(b - c)(a + d)(b + d) = q^2 - p^2.$$

163. When n is a positive integer, sum the series

$$1 + 3n + \frac{3 \cdot 4}{1 \cdot 2} \cdot \frac{n(n-1)}{1 \cdot 2} + \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + \ldots$$

164. If a, b, c are in A.P., G.P., or H.P., then $a^n + c^n > 2b^n$.

165. Prove that if the sum of two quantities is constant their product is greatest when they are equal. Use this theorem to prove that of all triangles with the same perimeter and the same base, the isosceles has the greatest area.

166. Using the fact that $(1 + x)^{2n} = (1 + 2x + x^2)^n$, prove

$$2^n + \frac{n(n-1)}{1^2} \cdot 2^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1^2 \cdot 2^2} \cdot 2^{n-4} + \ldots = \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

167. Show that

$$2^n - (n-1)2^{n-2} + \frac{(n-2)(n-3)}{1 \cdot 2} 2^{n-4} - \frac{(n-3)(n-4)(n-5)}{1 \cdot 2 \cdot 3} 2^{n-6} + \ldots = n + 1.$$

168. In how many ways can 18 be expressed as the sum of 8 odd numbers, when repetitions are allowed and the order of addition is taken into account?

169. If $2(a^3 + b^3 + c^3 + 6abc) = 3(a^2b + a^2c + b^2c + b^2a + c^2a + c^2b)$, then one of the quantities a, b, c is an arithmetic mean between the other two.

170. If the square of the sum of n real quantities is equal to $\frac{2n}{n-1}$ times the sum of their products two and two together, the quantities are all equal.

171. If a man 53 years of age can buy an annuity of \$300 a year for \$3625·60, interest being reckoned at 5%, determine what is considered the expectation of life at 53.

172. If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ and $x + y + z = 0$ are satisfied by two equal sets of ratios $x : y : z$, then

$$bc - f^2 + ca - g^2 + ab - h^2 + 2(gh - af + hf - bg + fg - ch) = 0.$$

173. Prove $1 + \left(\frac{n}{1}\right)^2 \cdot \frac{1}{3} + \left(\frac{n(n-1)}{1 \cdot 2}\right)^2 \cdot \frac{1}{3^2} + \left(\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\right)^2 \cdot \frac{1}{3^3} + \ldots$

$$= \left(\frac{4}{3}\right)^n \left(1 + \frac{n(n-1)}{1 \cdot 1} \cdot \frac{3}{4^2} + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 1 \cdot 2} \cdot \frac{3^2}{4^4} + \ldots\right).$$

174. If the square root of a number contains $2n + 1$ digits, show that when $n + 1$ digits have been found by the ordinary method the remaining n digits may be found by division.

175. If $x_n = x(x + 1)(x + 2) \ldots (x + n - 1)$, show that

$$(x + y)_n = x_n + nx_{n-1}y_1 + \frac{n(n-1)}{1.2}x_{n-2}y_2 + \ldots + y_n.$$

176. The number of combinations n at a time of $3n$ things, of which n are alike and the rest all different, is $2^{2n-1} + \lfloor 2n - 1 \div \lfloor n \rfloor n - 1$.

177. If x, y be unequal, and if $\frac{(2x - y - z)^3}{x} = \frac{(2y - z - x)^3}{y}$, then each fraction $= \frac{(2z - x - y)^3}{z}$.

178. If z, x, y are in G.P. when l is subtracted from each, and x, y, z are in G.P. when m is subtracted from each, and y, z, x are in G.P. when n is subtracted from each, then

$$\frac{1}{x - l} + \frac{1}{y - m} + \frac{1}{z - n} = 0.$$

179. If the equation $\frac{a}{x + a} + \frac{b}{x + b} = \frac{c}{x + c} + \frac{d}{x + d}$ has a pair of equal roots, show that one of the quantities a or b is equal to one of the quantities c or d or else

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} + \frac{1}{d}.$$

180. If $\frac{1}{1 + l + ln} + \frac{m}{1 + m + ml} + \frac{mn}{1 + n + mn} = 1,$

and $\frac{l}{1 + l + ln} + \frac{ml}{1 + m + ml} + \frac{1}{1 + n + mn} = 1,$

and none of the denominators be zero, then $l = m = n$.

181. Two straight lines intersect at a point A. On one of the lines are chosen p points and on the other q points. In the plane containing the lines there are chosen $r - 1$ additional points, none of which lies on either of the lines. Including the point A itself this makes a total of $p + q + r$ points. If, with the exception of the original pair of lines, no straight line can be drawn to pass through more than 2 of the $p + q + r$ points, find the number of triangles which can be formed with these points as angular points.

182. ABC, $A_1B_1C_1$, $A_2B_2C_2$, ... are an infinite sequence of congruent and similarly placed triangles, B_1 being the middle point of BC, B_2 of B_1C, B_3 of B_2C, etc. Show that the total area covered by the triangles (no part being counted more than once) is $2\frac{2}{3}$ of the area of ABC.

AMOUNT of 1 at interest, compounded yearly, for a given number of years.

Yrs.	1½%	2%	2½%	3%	3½%	4%	5%	6%
1	1·01500	1·02000	1·02500	1·03000	1·03500	1·04000	1·05000	1·06000
2	1·03023	1·04040	1·05063	1·06090	1·07123	1·08160	1·10250	1·12360
3	1·04568	1·06121	1·07689	1·09273	1·10872	1·12486	1·15763	1·19102
4	1·06136	1·08243	1·10381	1·12551	1·14752	1·16986	1·21551	1·26248
5	1·07728	1·10408	1·13141	1·15927	1·18769	1·21665	1·27628	1·33823
6	1·09344	1·12616	1·15969	1·19405	1·22926	1·26532	1·34010	1·41852
7	1·10985	1·14869	1·18869	1·22987	1·27228	1·31593	1·40710	1·50363
8	1·12649	1·17166	1·21840	1·26677	1·31681	1·36857	1·47746	1·59385
9	1·14339	1·19509	1·24886	1·30477	1·36290	1·42331	1·55133	1·68948
10	1·16054	1·21899	1·28008	1·34392	1·41060	1·48024	1·62889	1·79085
11	1·17795	1·24337	1·31209	1·38423	1·45997	1·53945	1·71034	1·89830
12	1·19562	1·26824	1·34489	1·42576	1·51107	1·60103	1·79586	2·01220
13	1·21355	1·29361	1·37851	1·46853	1·56396	1·66507	1·88565	2·13293
14	1·23176	1·31948	1·41297	1·51259	1·61869	1·73168	1·97993	2·26090
15	1·25023	1·34587	1·44830	1·55797	1·67535	1·80094	2·07893	2·39656
16	1·26899	1·37279	1·48451	1·60471	1·73399	1·87298	2·18287	2·54035
17	1·28802	1·40024	1·52162	1·65285	1·79468	1·94790	2·29202	2·69277
18	1·30734	1·42825	1·55966	1·70243	1·85749	2·02582	2·40662	2·85434
19	1·32695	1·45681	1·59865	1·75351	1·92250	2·10685	2·52695	3·02560
20	1·34686	1·48595	1·63862	1·80611	1·98979	2·19112	2·65330	3·20714
21	1·36706	1·51567	1·67958	1·86029	2·05943	2·27877	2·78596	3·39956
22	1·38756	1·54598	1·72157	1·91610	2·13151	2·36992	2·92526	3·60354
23	1·40838	1·57690	1·76461	1·97359	2·20611	2·46472	3·07152	3·81975
24	1·42950	1·60844	1·80873	2·03279	2·28333	2·56330	3·22510	4·04893
25	1·45095	1·64060	1·85394	2·09378	2·36324	2·66584	3·38635	4·29187
26	1·47271	1·67342	1·90029	2·15659	2·44596	2·77247	3·55567	4·54938
27	1·49480	1·70689	1·94780	2·22129	2·53157	2·88337	3·73346	4·82235
28	1·51722	1·74102	1·99650	2·28793	2·62017	2·99870	3·92013	5·11169
29	1·53998	1·77584	2·04641	2·35657	2·71188	3·11865	4·11614	5·41839
30	1·56308	1·81136	2·09757	2·42726	2·80679	3·24340	4·32194	5·74349
31	1·58653	1·84759	2·15001	2·50008	2·90503	3·37313	4·53804	6·08810
32	1·61032	1·88454	2·20376	2·57508	3·00671	3·50806	4·76494	6·45339
33	1·63448	1·92223	2·25885	2·65234	3·11194	3·64838	5·00319	6·84059
34	1·65900	1·96068	2·31532	2·73191	3·22086	3·79432	5·25335	7·25103
35	1·68388	1·99988	2·37320	2·81386	3·33359	3·94608	5·51601	7·68609
36	1·70914	2·03988	2·43253	2·89827	3·45026	4·10393	5·79181	8·14725
37	1·73478	2·08068	2·49334	2·98522	3·57102	4·26808	6·08140	8·63609
38	1·76080	2·12229	2·55568	3·07478	3·69601	4·43881	6·38547	9·15425
39	1·78721	2·16474	2·61957	3·16702	3·82537	4·61636	6·70475	9·70351
40	1·81402	2·20803	2·68506	3·26203	3·95925	4·80102	7·03998	10·28572

PRESENT VALUE of 1, due at the end of a given number of years, interest compounded yearly.

Yrs.	1½%	2%	2½%	3%	3½%	4%	5%	6%
1	·98522	·98039	·97561	·97087	·96618	·96154	·95238	·94340
2	·97066	·96117	·95181	·94260	·93351	·92456	·90703	·89000
3	·95632	·94232	·92860	·91514	·90194	·88900	·86384	·83962
4	·94218	·92385	·90595	·88849	·87144	·85480	·82270	·79209
5	·92826	·90573	·88385	·86261	·84197	·82193	·78353	·74726
6	·91454	·88797	·86230	·83748	·81350	·79031	·74622	·70496
7	·90103	·87056	·84127	·81309	·78599	·75992	·71068	·66506
8	·88771	·85349	·82075	·78941	·75941	·73069	·67684	·62741
9	·87459	·83675	·80073	·76642	·73373	·70259	·64461	·59190
10	·86167	·82035	·78120	·74409	·70892	·67556	·61391	·55840
11	·84893	·80426	·76214	·72242	·68495	·64958	·58468	·52679
12	·83639	·78849	·74356	·70138	·66178	·62460	·55684	·49697
13	·82403	·77303	·72542	·68095	·63940	·60057	·53032	·46884
14	·81185	·75788	·70773	·66112	·61778	·57748	·50507	·44230
15	·79985	·74302	·69047	·64186	·59689	·55526	·48102	·41727
16	·78803	·72845	·67363	·62317	·57671	·53391	·45811	·39365
17	·77639	·71416	·65720	·60502	·55720	·51337	·43630	·37136
18	·76491	·70016	·64117	·58739	·53836	·49363	·41552	·35034
19	·75361	·68643	·62553	·57029	·52016	·47464	·39573	·33051
20	·74247	·67297	·61027	·55368	·50257	·45639	·37689	·31181
21	·73150	·65978	·59539	·53755	·48557	·43883	·35894	·29416
22	·72069	·64684	·58087	·52189	·46915	·42196	·34185	·27751
23	·71004	·63416	·56670	·50669	·45329	·40573	·32557	·26180
24	·69954	·62172	·55288	·49193	·43796	·39012	·31007	·24698
25	·68921	·60953	·53939	·47761	·42315	·37512	·29530	·23300
26	·67902	·59758	·52624	·46369	·40884	·36069	·28124	·21981
27	·66899	·58586	·51340	·45019	·39501	·34682	·26785	·20737
28	·65910	·57438	·50088	·43708	·38165	·33348	·25509	·19563
29	·64936	·56311	·48866	·42435	·36875	·32065	·24295	·18456
30	·63976	·55207	·47674	·41199	·35628	·30832	·23138	·17411
31	·63031	·54125	·46512	·39999	·34423	·29646	·22036	·16426
32	·62099	·53063	·45377	·38834	·33259	·28506	·20987	·15496
33	·61182	·52023	·44270	·37703	·32134	·27409	·19987	·14619
34	·60277	·51003	·43191	·36605	·31048	·26355	·19036	·13791
35	·59387	·50003	·42137	·35538	·29998	·25341	·18129	·13011
36	·58509	·49022	·41109	·34503	·28983	·24367	·17266	·12274
37	·57644	·48061	·40107	·33498	·28003	·23430	·16444	·11579
38	·56792	·47119	·39129	·32523	·27057	·22529	·15661	·10924
39	·55953	·46195	·38174	·31575	·26141	·21662	·14915	·10306
40	·55126	·45289	·37243	·30656	·25257	·20829	·14205	·09722

Four-Place Logarithms of Numbers.

No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Four-Place Logarithms of Numbers.

No.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2		3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

Table of Mortality.

Age	Number Living	Number Dying	Yearly Probability of Dying	Age	Number Living	Number Dying	Yearly Probability of Dying
10	100 000	749	·007 490	53	66 797	1 091	·016 333
11	99 251	746	·007 516	54	65 706	1 143	·017 396
12	98 505	743	·007 543	55	64 563	1 199	·018 571
13	97 762	740	·007 569	56	63 364	1 260	·019 885
14	97 022	737	·007 596	57	62 104	1 325	·021 335
15	96 285	735	·007 634	58	60 779	1 394	·022 936
16	95 550	732	·007 661	59	59 385	1 468	·024 720
17	94 818	729	·007 688	60	57 917	1 546	·026 693
18	94 089	727	·007 727	61	56 371	1 628	·028 880
19	93 362	725	·007 765	62	54 743	1 713	·031 292
20	92 637	723	·007 805	63	53 030	1 800	·033 943
21	91 914	722	·007 855	64	51 230	1 889	·036 873
22	91 192	721	·007 906	65	49 341	1 980	·040 129
23	90 471	720	·007 958	66	47 361	2 070	·043 707
24	89 751	719	·008 011	67	45 291	2 158	·047 647
25	89 032	718	·008 065	68	43 133	2 243	·052 002
26	88 314	718	·008 130	69	40 890	2 321	·056 762
27	87 596	718	·008 197	70	38 569	2 391	·061 993
28	86 878	718	·008 264	71	36 178	2 448	·067 665
29	86 160	719	·008 345	72	33 730	2 487	·073 733
30	85 441	720	·008 427	73	31 243	2 505	·080 178
31	84 721	721	·008 510	74	28 738	2 501	·087 028
32	84 000	723	·008 607	75	26 237	2 476	·094 371
33	83 277	726	·008 718	76	23 761	2 431	·102 311
34	82 551	729	·008 831	77	21 330	2 369	·111 064
35	81 822	732	·008 946	78	18 961	2 291	·120 827
36	81 090	737	·009 089	79	16 670	2 196	·131 734
37	80 353	742	·009 234	80	14 474	2 091	·144 466
38	79 611	749	·009 408	81	12 383	1 964	·158 605
39	78 862	756	·009 586	82	10 419	1 816	·174 297
40	78 106	765	·009 794	83	8 603	1 648	·191 561
41	77 341	774	·010 008	84	6 955	1 470	·211 359
42	76 567	785	·010 252	85	5 485	1 292	·235 552
43	75 782	797	·010 517	86	4 193	1 114	·265 681
44	74 985	812	·010 829	87	3 079	933	·303 020
45	74 173	828	·011 163	88	2 146	744	·346 692
46	73 345	848	·011 562	89	1 402	555	·395 863
47	72 497	870	·012 000	90	847	385	·454 545
48	71 627	896	·012 509	91	462	246	·532 466
49	70 731	927	·013 106	92	216	137	·634 259
50	69 804	962	·013 781	93	79	58	·734 177
51	68 842	1 001	·014 541	94	21	18	·857 143
52	67 841	1 044	·015 389	95	3	3	1·000 000

ANSWERS
TO
SENIOR HIGH SCHOOL ALGEBRA

ANSWERS

Exercise 1 (page 4)

1. 4. **2.** $\frac{7}{2}\frac{9}{9}$. **3.** abc. **4.** $\frac{a(1-b^2)}{b(a^2-1)}$. **5.** $\frac{1}{13}$, $\frac{1}{65}$.

6. 3, 11. **7.** $a \pm b$. **8.** 0, $\frac{1}{2}$, $\frac{1}{3}$. **9.** 1, $\frac{1}{2}(-1 \pm \sqrt{-3})$.

10. -3, $\frac{1}{2}(3 \pm 3\sqrt{-3})$. **11.** ± 1, $\pm \sqrt{-1}$.

12. ± 1, $\frac{1}{2}(\pm 1 \pm \sqrt{-3})$. **13.** $-a$, $-b$. **14.** 0, $\frac{2ab - ac - bc}{a + b - 2c}$.

15. 1, $1 \pm \frac{3}{2}\sqrt{2}$. **16.** -1, $\frac{1}{2}(1 \pm \sqrt{4n-3})$.

17. 0, $-.2$, $\pm 2\sqrt{2}$, $1 \pm \sqrt{-3}$. **18.** 0, ± 3, $\frac{1}{2}(3 \pm 3\sqrt{-3})$.

19. $\frac{cn + bd}{an + bm}$, $\frac{cm - ad}{an + bm}$. **20.** $b + c$, $c + a$. **21.** (3, 2), $(-8, 13)$.

22. (3, 2), (1, 1). **23.** (7, 2), (2, 7). **24.** (2, 3), $(1\frac{1}{2}, 4)$.

25. $(-1, 5)$, $(5, -25)$, $(-1 \pm \sqrt{6}, -1 \pm \sqrt{6})$.

26. $(\pm 7, \pm 5)$, $(\pm 1, \mp 1)$. **27.** $(\pm 4, \mp 2)$, $(\pm \frac{1}{2}, \pm 5)$.

28. $(\pm 1, \pm 2)$, $(\pm \frac{7}{2\,2}\sqrt{22}, \mp \frac{1}{2\,2}\frac{3}{2}\sqrt{22})$. **29.** $(-4, 2)$, $(0, 0)$, $(1\frac{5}{7}, \frac{4}{7})$.

30. (1, 3), $(\sqrt[3]{5}, \sqrt[3]{5})$, $(\sqrt[3]{\frac{5}{2}}, 2\sqrt[3]{\frac{5}{2}})$. **31.** $(\pm 7, \mp 5)$, $(\pm 5, \mp 7)$.

32. (4, 1), (1, 4).

Exercise 2 (page 7)

1. 3. **2.** $1\frac{1}{2}$. **3.** 4, $-3\frac{1}{3}$. **4.** 5. **5.** 1. **6.** $1, 2, \frac{3}{2} \pm \frac{1}{2}\sqrt{145}$.

7. 2, -5. **8.** 2, $-\frac{1}{3}$, $\frac{1}{2}$, $1\frac{1}{6}$. **9.** 13. **10.** $2, \frac{1}{2}, \frac{1}{4}(5 \pm \sqrt{201})$.

11. ± 2, ± 5. **12.** $1, -2, \frac{1}{2}(-1 \pm \sqrt{-39})$. **13.** $3, \frac{1}{3}, \frac{1}{2\,0}(3 \pm \sqrt{-391})$.

Exercise 2 (page 7) *continued*

14. $2, \frac{1}{2}, \frac{1}{5}(1 \pm 2\sqrt{-6})$. **15.** $5, -\frac{1}{2}, \frac{1}{4}(9 \pm \sqrt{-111})$.

16. $1, -6\frac{1}{2}, \frac{1}{4}(-11 \pm \sqrt{-199})$. **17.** $\frac{2}{3}, \frac{3}{2}, \frac{1}{2}(-1 \pm \sqrt{-3})$.

18. $\frac{3}{4}, \frac{4}{3}, \frac{1}{2}(-1 \pm \sqrt{-3})$. **19.** $2 \pm \sqrt{3}, 3 \pm 2\sqrt{2}$.

20. $2, -\frac{1}{2}, 3, -\frac{1}{3}$. **21.** $(2, 3), (3, 2)$.

22. $(12, 3), (3, 12)$. **23.** $(2, 1), (-1, -2), (\pm\sqrt{3}, \pm\sqrt{3})$.

24. $(\pm 5, \pm 2), (\pm 2, \pm 5)$. **25.** $(15, 11), (-11, -15)$.

26. $(4, 5), (5, 4), (-5\frac{1}{2} \pm \frac{1}{2}\sqrt{-39}, -5\frac{1}{2} \mp \frac{1}{2}\sqrt{-39})$.

27. $(4, 12), (12, 4), (-5 \pm \sqrt{55}, -5 \mp \sqrt{55})$. **28.** $(\frac{1}{10}, \frac{1}{6}), (\frac{1}{5}, \frac{1}{10})$.

29. $(27, 8)$. **30.** $(4, 1), (2, 2), (-3, -1\frac{1}{3}), (-6, -\frac{2}{3})$.

31. $(4, 3), (3, 4), (-1 \pm \sqrt{7}, -1 \mp \sqrt{7})$. **32.** $(1, 3), (3, 1)$.

33. $(6, 2), (2, 6), (4 \pm 10\sqrt{-1}, 4 \mp 10\sqrt{-1})$.

34. $(5, 1), (-1, -5), (2 \pm \sqrt{-33}, -2 \pm \sqrt{-33})$. **35.** $10, 11, 12, 13$.

36. $3, 5, 7, 9$. **37.** $1, \frac{1}{2}(a \pm \sqrt{a^2 - 4})$, where $a = \frac{1}{2}(-1 \pm \sqrt{5})$.

Exercise 3 (page 10)

1. $16, 12, -10$. **2.** $1, 1\frac{1}{2}, 2$. **3.** $1\frac{5}{7}, 2\frac{2}{5}, -12$.

4. $b + c - a, \quad c + a - b, \quad a + b - c$. **5.** $(0, 0, 0), (1\frac{1}{2}, 2, 3\frac{1}{2})$.

6. $\pm 5, \pm 7, \pm 1$. **7.** $\pm 12, \pm 6, \pm 8$. **8.** $(4, 2, 1), (7\frac{5}{17}, -21\frac{1}{17}, -12\frac{8}{17})$.

9. $(1, 2, 4), (1, 4, 2), (6, \frac{1}{2} \pm \frac{1}{2}\sqrt{-31}, \frac{1}{2} \mp \frac{1}{2}\sqrt{-31})$.

10. $(2, 8, 4), (8, 2, 4)$. **11.** $\pm 2, \pm 8, \pm \frac{1}{2}$. **12.** $\pm 3, \pm 2, \pm 5$.

13. $\pm 2, \pm 3, \pm 1$. **14.** $\pm (b + c - a), \pm (c + a - b), \pm (a + b - c)$.

15. $\pm 3, \pm 6, \pm 1$. **16.** $\frac{a(b-c)^2}{(a-b)(c-a)}, \frac{b(c-a)^2}{(b-c)(a-b)}, \frac{c(a-b)^2}{(c-a)(b-c)}$.

17. $(2, 3, 1), (3, 2, 1)$. **18.** $(5, 6, 7), (-1, -2, -3)$. **19.** $\pm 1, \mp 2, \mp 4$.

20. $(5, 1, 1), (1, 5, 1), (1, 1, 5), (2, 2, 2), (-3, -3, -3)$.

21. $\frac{1}{3}(2a - b - c), \frac{1}{3}(2b - c - a), \frac{1}{3}(2c - a - b)$.

Exercise 4 (page 15)

1. 23, 58. **2.** $-9, -44$. **3.** $4\frac{1}{2}, 11\frac{1}{2}$. **4.** $-2\frac{1}{2}, -13$.
5. $9a, 23a$. **6.** $-\cdot4, -3\cdot2$. **7.** $a-3b, a-10b$. **8.** $6a-5b, 13a-12b$.
9. $5x+9, 12x+23$. **10.** $-3(m+n), -10(m+n)$. **11.** 199, $93\frac{1}{2}$.
12. $10\sqrt{2}, 5\sqrt{5}$. **13.** n. **14.** $2n-1$. **15.** $3n+4$.
16. $5-3n$. **17.** $5x-2nx$. **18.** 1. **19.** $a^2+4ab+b^2-2abn$.
20. 0. **21.** 20th. **22.** 100th. **23.** 15th. **24.** 23rd.
25. 4. **26.** 260. **27.** 40. **28.** -110. **29.** 26th. **30.** 91.
31. 57. **32.** 3, 4. **33.** 6. **34.** 123. **35.** 64. **38.** 10.
39. 3, 5, 7. **40.** 46. **43.** 15, 20. **45.** 2500.

Exercise 5 (page 19)

1. 1600. **2.** 400. **3.** -400. **4.** $169a$. **5.** $2nx$. **6.** $2an$.
7. 590. **8.** 372. **9.** -2592. **10.** $265\frac{1}{2}$. **11.** $-7\frac{1}{2}$.
12. 99. **13.** $22\frac{2}{3}$. **14.** -84. **15.** $144a$. **16.** 595. **17.** $\frac{3}{2}n^2+\frac{1}{2}n$.
18. $\frac{1}{2}n^2+\frac{1}{2}n$. **19.** $2n^2$. **20.** $3n^2+2n$. **21.** $4n-n^2$.
22. $5\frac{1}{4}n-1\frac{1}{4}n^2$. **23.** $2an^2-an$. **24.** $an+n^2$. **25.** $\frac{1}{2}n-\frac{1}{2}$.
26. $n(a^2+3ab+b^2-anb)$. **27.** n^2. **29.** 2120. **30.** 0.
32. 8 or 9. **33.** 15 or 22. **34.** 17. **35.** n^2+n. **36.** 1, 3, 5, . . .
37. 17. **38.** 1560 yd. **40.** 250500. **41.** 247500. **42.** $2n+2$.
43. 3340 **44.** 2318·4 ft. **45.** 8. **46.** 7. **47.** 32. **48.** 19th.

Exercise 6 (page 22)

1. 18, -26, a, $4x+y$. **2.** $d=6$; $\frac{2}{3}$; $1\frac{1}{2}$; $-a$; $x-1$.
3. $d=3$, $s=670$. **4.** 313. **5.** $x+\dfrac{r(y-x)}{m+1}$. **6.** 1, -2.
7. $\frac{1}{2}n(a+b)$. **10.** $\frac{1}{3}(2a+b)$, $\frac{1}{3}(a+2b)$. **12.** 12.

Exercise 7 (page 26)

2. $n^2 - 2n$. **3.** $4n + 1$. **4.** $(a - b) + (3a - b) + \ldots$
5. $\frac{1}{2}n(3n + 7)$. **9.** 7, 9, 11. **13.** $\frac{1}{4}(3a + b)$, $\frac{1}{2}(a + b)$, $\frac{1}{4}(a + 3b)$.
14. 12 or -9. **15.** $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, . . . **17.** 45. **18.** $-2n$, $2n + 1$.
20. $\frac{(a + c)(b + c - 2a)}{2(b - a)}$. **21.** $\frac{1}{10}$, $\frac{1}{5}$, $\frac{1}{10}$, $\frac{2}{5}$, **22.** 6, 4, 2, 0, -2.
24. 2, 4, 6, 8. **27.** $\frac{1}{4}n^2(n + 1)^2$, n^4. **28.** 1, 4, 7, 10, . . . **29.** 9.
31. $-p - q$. **33.** 12 days. **34.** 7 or -11.

Exercise 8 (page 32)

1. 512. **2.** $\frac{1}{81}$. **3.** $1\frac{5}{27}$. **4.** $-\frac{5}{16}$. **5.** x^{19}. **6.** $-\frac{1}{64}$
7. $a^{-8}b^{10}$. **8.** $a^{21}b^{20}$. **9.** x^{-4}, x^{3-n}. **10.** -2^{1-2n}, 2^{-2n}.
11. $(a + b)^{n-1}(a - b)^{3-n}$. **12.** $\frac{1}{3}$. **13.** $a = 10$, $r = \pm 2$. **14.** $-1\frac{1}{28}$.
15. 2. **17.** 8th, 6th. **18.** 6; 27; 2; $a^2 - b^2$. **19.** 15, 45.
20. $r = 2$; $\frac{2}{3}\sqrt{3}$; y. **21.** 192. **22.** $\frac{1}{2}$, 2, 8. **25.** 4, 16.
26. 5, 125. **31.** 3, 6, 12. **32.** 2, 6, 18. **35.** 18, 12.

Exercise 9 (page 35)

1. 511. **2.** $15\frac{1}{5}\frac{1}{2}$. **3.** $10\frac{5}{8}$. **4.** $10\frac{1}{4}\frac{1}{6}$. **5.** $40(\sqrt{3} + 1)$.
6. ·111111. **7.** $\frac{1}{3}(1 - 2^{2n})$. **8.** $\frac{1}{3}(2 + 2^{-2n})$.
9. $\frac{a}{2a - 1}(2^n a^n - 1)$. **10.** 20·0235. **11.** 8190. **12.** 2.
13. $46\frac{8}{9}$, $12\frac{2}{9}$. **15.** 2. **16.** $\frac{(1 + a)(a^n b^n - 1)}{ab - 1}$.

Exercise 10 (page 38)

1. 8. **2.** $1\frac{1}{2}$. **3.** $\frac{2}{3}$. **4.** 15. **5.** $4\frac{1}{2} + \frac{3}{2}\sqrt{3}$. **6.** $\frac{5}{9}$.
7. $\frac{4}{3}a$. **8.** $\frac{1}{2}(25 + 5\sqrt{15})$. **9.** 21. **10.** 41. **11.** $\frac{a}{1 + r}$.
12. $\frac{1}{5}$. **13.** $\frac{4}{9}$. **14.** $\frac{99}{11}\frac{9}{10}$. **15.** $\frac{2}{3}$, $\frac{14}{99}$, $\frac{148}{990}$. **17.** 1, $\frac{3}{5}$, $\frac{9}{25}$, . . .
18. 4, 1, $\frac{1}{4}$, . . . **20.** $\frac{1}{14}\frac{1}{55}$. **22.** 24, -8, $2\frac{2}{3}$, . . .
23. $a = 1\frac{3}{8}$ or $6\frac{3}{8}$, $r = \pm\frac{3}{5}$. **25.** $x < 2\frac{1}{2}$ and $> -2\frac{1}{2}$. **26.** 224 ft.
32. $\frac{3}{3} + \frac{4}{9}x + \frac{8}{27}x^2 + \ldots$, where $x < 1\frac{1}{2}$ and $> -1\frac{1}{2}$.

Exercise 11 (page 45)

1. 15, 12. **2.** $\frac{3}{5}$, $\frac{1}{2}$. **3.** 1, $\frac{4}{5}$. **4.** 12, 24.

5. $-1\frac{5}{7}$, $-1\frac{1}{5}$. **6.** $\frac{1}{7}(3-\sqrt{2})$, $\frac{1}{14}(4-\sqrt{2})$.

7. 1. **8.** $\frac{4}{9-n}$. **9.** $\frac{ab}{b+(n-1)(a-b)}$. **10.** 840, 420.

11. 16th. **12.** 20, -24, $\frac{1}{8}$, $\frac{a^2-b^2}{a^2+b^2}$.

13. $1\frac{1}{7}$, $1\frac{1}{3}$, $1\frac{3}{5}$; $3\frac{1}{2}$, $1\frac{2}{5}$, $\frac{7}{8}$, $\frac{7}{11}$; $1\frac{1}{2}$, 1, $\frac{3}{4}$, . . . $\frac{3}{11}$. **14.** $\frac{3ab}{a+2b}$, $\frac{3ab}{2a+b}$.

15. $\frac{ab(n+1)}{b(n+1)+(a-b)}$, $\frac{ab(n+1)}{b(n+1)+2(a-b)}$, . . . **16.** 147.

19. 20, 12 or 3, -5. **20.** 10, 15. **21.** 30, 20.

22. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$ or $\frac{7}{24}$, $\frac{1}{2}$, $\frac{7}{4}$, $-\frac{7}{6}$, $-\frac{7}{16}$. **23.** 3.

24. $\frac{ab+bc-2ac}{a+c-2b}$. **26.** $\frac{5}{4}$, $\frac{5}{3}$, $\frac{5}{2}$, 5, ∞, -5, $-\frac{5}{2}$, $-\frac{5}{3}$, $-\frac{5}{4}$.

Exercise 12 (page 48)

1. 2, 4. **2.** 19·2. **6.** 4, 8, 16, 32.

14. $\frac{5}{8}$, 1, $\frac{8}{5}$ or $-\frac{3}{8}$, -1, $-\frac{8}{3}$.

16. $2a+3b-\frac{a}{2^{19}}-\frac{b\,.\,2^{20}}{3^{19}}$, $2a+3b$. **17.** $(a-br)\frac{1-r^{2n}}{1-r^2}$.

18. $\frac{1}{2}n(n+1)+\frac{1}{9}(1-10^{-n})$; $n-\frac{1}{9}(1-10^{-n})$.

19. $\frac{1}{2}n(n+1)+1-2^{-n}$; $2^{n+1}-2-n$.

20. $\frac{r^{2n+2}-r^2}{r^2-1}+\frac{r^{2n}-1}{r^{2n}(r^2-1)}-2n$. **30.** $1\frac{1}{2}$, 2, 3.

Exercise 13 (page 54)

1. 18, $\frac{2}{15}$, $\frac{bc}{a}$, $(a+b)(a-b)^2$. **2.** 9, $\frac{1}{6}$, 1, 3.

3. 36, $(x-y)(x+y)^2$. **4.** $8\frac{3}{5}$. **5.** 13. **6.** 36, 42.

7. 2. **8.** 30, 45, 75, 90. **9.** $\frac{-ab}{a+b}$. **10.** 10, 15.

Exercise 14 (page 59)

3. The latter. **4.** The latter. **5.** $2 : 3$.

6. $d - b : a - c$. **7.** $3 : 1$ or $1 : 3$. **8.** $1 : 2$ or $4 : 3$.

9. $3 \pm \sqrt{29} : 10$. **10.** $-3 : 2 : 1$. **11.** $2 : 1 : 3$.

12. $b - c : c - a : a - b$. **13.** $2 : 1 : 4$. **14.** $5 : 3 : 2$.

15. $b - c : c - a : a - b$. **19.** $5 : 4 : 2$.

20. $a(b - c)^2 : b(c - a)^2 : c(a - b)^2$.

21. $\pm 4, \pm 6, \pm 8$. **22.** $\pm 1, \pm 1, \pm 4$. **23.** $5, 9, \frac{1}{5}$.

24. 18, 12, 6. **25.** 10, 16, 7. **26.** $b - a, c - b, a - c$.

27. $3, 4, 5$ or $-\frac{36}{91}, -\frac{48}{91}, -\frac{60}{91}$. **28.** $b - c, c - a, a - b$.

29. $\frac{1}{2}(b + c), \frac{1}{2}(c + a), \frac{1}{2}(a + b)$.

Exercise 15 (page 62)

1. $16a + 12b + 9c = 0$. **2.** $a^2 + b^2 = c(b^2 - a^2)$.

3. $5c = \pm 24d$. **4.** $a^3 + b^3 + c^3 = 3abc$.

5. $ab + bc + ca + 2abc = 1$.

6. $(bn - cm)^2 + (cl - an)^2 = 2(am - bl)^2$.

7. $(c - c_1)^2 = (bc_1 - b_1c)(b_1 - b)$.

8. $(ca_1 - c_1a)^3 = (bc_1 - b_1c)(ab_1 - a_1b)^2$.

Exercise 16 (page 66)

7. $\frac{-c}{a + b}, \frac{2ac}{c^2 + 1}$. **8.** $0, \pm\sqrt{6}, \pm\frac{1}{2}\sqrt{6}, \pm 2\sqrt{-1}$.

11. 5, 4, 3. **17.** $mq : np$.

Exercise 17 (page 72)

1. $a = mb$; $y = \frac{m}{x}$; $A = mR^2$; $s = mt^2$; $y^3 = m \div \sqrt{x}$.

2. V varies as the cube of r; r varies as the cube root of V.

3. 25, 225. **4.** $xy^2 = 48a^3$. **5.** $14xy = 14xz + 25yz$.

6. $\pm$ 1·25. **9.** 62½ lb. **13.** 17.

14. $2\frac{2}{5}$ or $-\frac{2}{5}$. **15.** $86\frac{5}{8}$ sq. ft., $3\frac{1}{2}$ ft.

16. 1600 ft.; 16 ft., 304 ft., $10\sqrt{2}$ sec. **17.** $1\frac{1}{9}$. **18.** 6 in.

19. $2{\cdot}25m$. **20.** 12·8 ft. **21.** 11·86 yr. nearly. **22.** 1504.

23. \$650. **24.** \$288. **25.** \$147·50.

Exercise 18 (page 76)

3. $6\frac{3}{4}$. **4.** $y = 3 + 2x + x^2$. **9.** 60 cu. ft. **10.** $67\frac{1}{2}$.

11. 21 ohms. **12.** 6·28 + lb. **13.** 9 sq. ft. **14.** $1\frac{1}{3}$ lb.

15. 3360 lb. **17.** 8 : 5. **18.** $V = \frac{1}{3}bh$. **19.** $\frac{3}{7}$, $\frac{4}{7}$.

20. 143. **21.** \$3·60. **22.** 20 : 1.

Exercise 19 (page 81)

1. $\frac{1}{6}(7\sqrt{2} + \sqrt{3})$. **2.** $26 - 15\sqrt{3}$. **3.** $1 + \sqrt{2} + \sqrt{3}$.

4. $1 + \sqrt{5} + \sqrt{10} - \sqrt{2}$. **5.** $\frac{1}{7}(3\sqrt{30} + 5\sqrt{15} - 10\sqrt{2} - 12)$.

6. $\frac{1}{2ab}(a\sqrt{b} + b\sqrt{a} + \sqrt{a^2b + ab^2})$. **7.** ·2679492. **8.** $\sqrt{3}$.

9. ·175. **10.** 15. **11.** 10. **12.** 1. **13.** 289. **14.** $\frac{1}{2}a - \frac{1}{2}$.

Exercise 20 (page 83)

1. $3\sqrt{3} - \sqrt{6}$. **2.** $2\sqrt{13} + 7$. **3.** $\sqrt{2} - \frac{1}{6}$. **4.** $\sqrt{a+b} - \sqrt{a-b}$.

5. $a - b - 2\sqrt{ab}$. **6.** $\sqrt{\frac{1}{2}(3a+b)} - \sqrt{\frac{1}{2}(a-b}$.

7. $\sqrt{\frac{1}{2}(1 + a + a^2)} + \sqrt{\frac{1}{2}(1 - a + a^2)}$. **8.** $\sqrt{5} + 1$, $\sqrt{7} - \sqrt{5}$.

9. $\sqrt{2} - 1$, $3\sqrt{3} + 5$. **10.** 1. **14.** 1, -7, $3 \pm \sqrt{5}$.

15. $2 - \sqrt{3}$. **16.** $3 - \sqrt{5}$, $3 + 2\sqrt{2}$. **17.** 4·657.

18. $1 + \sqrt{2} + \sqrt{3}$, $3 - 2\sqrt{2} + \sqrt{5}$.

Exercise 21 (page 87)

1. $5, 5\omega, 5\omega^2$; $\frac{1}{2}a, \frac{1}{2}a\omega, \frac{1}{2}a\omega^2$.

2. $(x-y)(x-\omega y)(x-\omega^2 y)$; $\left(a+\frac{2}{b}\right)\left(a+\frac{2\omega}{b}\right)\left(a+\frac{2\omega^2}{b}\right)$;
$(x-\omega y)(x-\omega^2 y)$; $(x-y-z)(x-\omega y-\omega^2 z)(x-\omega^2 y-\omega z)$.

14. $\dfrac{3(a^2-bc)}{a^3+b^3+c^3-3abc}$.

Exercise 22 (page 92)

1. Real and irrational; real and rational; imaginary; real and equal.

2. $\pm 12, \frac{3}{8}, 4, 6$ or $-\frac{3}{4}$.

3. $(4x-9)(5x-12)$, $(47x+1)(23x-1)$,
$\{(a-b)x+2a-b\}\{(2a-b)x-3a+b\}$.

4. $\frac{2}{3}, \frac{2}{5}, -2, 11\frac{5}{9}$.

5. $x^2-10x+16=0$, $x^2-4x-2=0$.

8. 2.

11. $5, -\frac{1}{3}$.

14. $3pq-p^3$, $p^4-4p^2q+2q^2$.

15. $x^2-3px+2p^2+q=0$.

16. $ac, \dfrac{b^2-2ac}{a^2c^2}$.

17. $5b^2=36ac$.

18. $4m, 4n$.

19. $6, -2$.

20. $x^2-x(a^3-3ab)+b^3=0$.

21. $3ax^2-3a^2x+a^3-b^3=0$.

22. $6\frac{1}{2}, -3\frac{1}{2}$.

23. $7, -1$.

24. $rpx^2-x(q^2-2pr)+pr=0$.

25. $bx^2-2x+a=0$.

27. $4x^2-24x+16$.

28. $64x^2-36x+5=0$ or $64x^2-4x-5=0$.

29. $x^2-4abx-(a^2-b^2)^2=0$.

34. a, b, c all have the same sign; c differs in sign from b and a.

Exercise 23 (page 97)

1. Between 2 and $3\frac{1}{2}$; between $5\frac{1}{2}$ and $-\frac{2}{3}$; between $2+\sqrt{5}$ and $2-\sqrt{5}$.

2. Between 7 and -3; not between 3 and $2\frac{1}{2}$; between $3+\sqrt{5}$ and $3-\sqrt{5}$.

4. $\frac{2}{5}, 3\frac{4}{13}$; between $\frac{2}{5}$ and $3\frac{4}{13}$; not between $\frac{2}{5}$ and $3\frac{4}{13}$.

7. 5.

8. 4, 5.

10. Between 4 and 7.

Exercise 24 (page 102)

1. 4, − 10. **2.** ± 1. **3.** $(b - q)^2 = (aq - bp)(p - a)$.

4. 0, 9; $x, x + 3$. **6.** 5, 2; 5, 4 or 0, 7; 0, 9.

7. $\frac{b(b-c)}{a(c-a)}$ or $\frac{c(c-a)}{b(a-b)}$. **10.** 13. **11.** − 5, 7.

12. $12abc(a + b + c)$. **15.** $5(a^2 + b^2 + c^2 - ab - bc - ca)$.

16. $5(a^2 + b^2 + c^2 + ab + bc + ca)$. **17.** 326, 13. **18.** 10, 40.

19. 2, 2, $-1\frac{1}{2}$. **20.** $dx^3 + cx^2 + bx + a = 0$.

21. $x^3 - 8x^2 + 20x - 24 = 0$.

22. (2, 3, 4), (2, 4, 3), (3, 2, 4), (3, 4, 2), (4, 2, 3), (4, 3, 2).

23. (3, 4, 5), etc. **24.** (3, − 1, − 2), etc. **25.** (2, 3, − 1), etc.

26. (6, − 2, − 3), etc. **28.** 1.

30. $\frac{f(a) - f(b)}{a - b}x + \frac{af(b) - bf(a)}{a - b}$. **31.** $ad = bc$. **32.** 8.

Exercise 25 (page 108)

1. Greater than 5, less than 5, greater than − 7, less than 7.

2. $x = y = 0$, $a = b = 0$, $x = 1$ and $y = 2$, $x = 1$ or $y = 2$.

3. 14, 1, $\frac{1}{20}$. **4.** 3, $\frac{1}{4}$, 4. **5.** $-2\frac{3}{4}$. **9.** 10, 100.

15. 400 sq. in. **16.** 64 in. **17.** $x = 3, y = -2$.

18. $x = 4, y = 3$. **23.** $-\frac{b}{2a}$. **29.** $\frac{a}{a^2 + b^2}, \frac{b}{a^2 + b^2}$.

32. $n, n + 1$.

Exercise 26 (page 111)

1. Not between 8 and 12. **2.** Between $\frac{1}{2}$ and $1\frac{1}{2}$.

3. Not between 1 and − 3. **4.** Between $\frac{1}{2}$ and $-\frac{1}{6}$.

5. Between $2\frac{1}{3}$ and − 3. **6.** Not between b and c.

7. 7, $\frac{1}{7}$. **10.** $1\frac{1}{3}$. **16.** 6, 9.

Exercise 27 (page 115)

1. $\frac{1+x}{(1-x)^2}$. **2.** $\frac{2(1-a^n)}{(1-a)^2} - \frac{1+(2n-1)a^n}{1-a}$.

3. $4 - \frac{1}{2^{n-2}} - \frac{n}{2^{n-1}}$, 4. **4.** $\frac{20}{9} - \frac{5+6n}{9 \cdot 4^{n-1}}$, $\frac{20}{9}$.

5. $\frac{r-r^{n+1}}{(1-r)^2} - \frac{nr^{n+1}}{1-r}$. **6.** $\frac{1}{(1-x)(1-2x)}$.

7. $\frac{1-r^n}{(1-r)(1-x)} - \frac{x(1-r^nx^n)}{(1-rx)(1-x)}$, if r and rx are <1, $\frac{1}{(1-r)(1-rx)}$.

8. $\frac{2x(1-x^n)}{(1-x)^2} - \frac{(2n+1)x^{n+1}-x}{1-x}$. **10.** $\frac{1+x}{(1-x)^3}$.

11. $\frac{2}{9} + \frac{6n+1}{9(-2)^{n-1}}$, $\frac{2}{9}$. **13.** \$1165, \$4410.

Exercise 28 (page 118)

1. 2870, 44100. **2.** 328065. **3.** $\frac{1}{3}n(n+1)(n+2)$.

4. $\frac{1}{3}n(4n^2-1)$. **5.** 166485. **6.** $\frac{1}{4}n(n+1)(n+2)(n+3)$.

7. $\frac{1}{3}n(n+1)(9n^2+n-4)$.

8. $\frac{1}{3}n(n+1)(2n-1)$, $\frac{1}{4}n(n+1)(n^2+n-10)$, $\frac{1}{4}n(n+1)(n^2+n-3)$.

9. 13. **10.** $3n^2+3n-3$. **11.** 78^2-66^2. **12.** $n^2(2n^2-1)$.

13. $na^2 + abn(n+1) + \frac{1}{6}nb^2(n+1)(2n+1)$.

14. $\frac{1}{6}n(n+1)(n+2)$. **16.** $na^2 + abn(n-1) + \frac{1}{6}nb^2(n-1)(2n-1)$.

17. $n^2(n+1)^2 - 2^{n+1} + 2$. **19.** $-n(2n+1)$, $(n+1)(2n+1)$.

20. $\frac{1}{12}n(n+4)(2n+1)$, $\frac{1}{12}(n+1)(2n^2+n+3)$.

21. $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

Exercise 30 (page 124)

1. 20. **2.** *ab, ac, ad, bc, bd, cd, ba, ca, da, cb, db, dc.* **3.** 4.

4. 72. **5.** 25. **6.** 30. **7.** 200. **8.** 12. **9.** 24.

10. 13800. **11.** 60. **12.** 6840, 8000. **13.** 720.

Exercise 31 (page 128)

1. 210, 840, 720. **3.** 720. **4.** 5040. **5.** 40320. **6.** 360.
7. 7. **8.** 7. **10.** 4. **11.** 60, 325. **12.** 720.
13. $\underline{|n}$, $(n-2)\underline{|n-1}$. **14.** 86400. **15.** 40320, 5040.
16. 504, 729. **17.** 696. **18.** 24. **19.** 1554. **20.** 720.
21. 3999960.

Exercise 32 (page 131)

1. 30240, 420, 1663200. **2.** 420. **3.** 12. **4.** 60. **5.** 1260.
6. 10, 50. **7.** 120, 60, 24, 6, 18. **8.** 4. **9.** 792. **10.** 27720.
11. 360. **12.** 126. **13.** 120, 420. **14.** 126. **15.** 240.

Exercise 33 (page 135)

1. 10, 28, 4845. **2.** 20, 455, 1287, 120. **3.** 6, 15, 20, 15, 6, 1.
4. 220. **5.** 6. **6.** 8. **7.** 7. **8.** 136, 1140. **9.** 99.
10. 28. **11.** 20. **12.** $\frac{1}{2}n(n-3)$. **13.** 11. **14.** $\frac{1}{2}n(n-1)$.
15. $\frac{1}{6}n(n-1)(n-2)$. **16.** 4950. **17.** 252. **18.** 1260.
19. 32318.

Exercise 34 (page 139)

1. 7056. **2.** 1092000, 218400. **3.** 495, 165, 330. **4.** 792.
5. 715, 1716, 1287. **6.** 5400. **7.** 2211 **8.** 11256.
10. 2639. **11.** 1023, 1013. **13.** 15. **14.** 41. **15.** 4095.
16. 22, 34 (not counting the number and unity).

Exercise 35 (page 143)

1. 10, 8 or 9, n, n or $n+1$. **2.** 50 cents.
4. $\underline{|20} \div \underline{|10}\,\underline{|10}$, $\frac{1}{2}\underline{|20} \div \underline{|10}\,\underline{|10}$. **5.** $\underline{|m+n} \div \underline{|m}\,\underline{|n}$.
6. $\underline{|52} \div (\underline{|13})^4$, $\frac{1}{24}\underline{|52} \div (\underline{|13})^4$. **7.** 7, 72; 46, 2275.
8. $\underline{|na} \div (\underline{|a}\,)^n$.

Exercise 36 (page 144)

1. 1900. **2.** 39800. **3.** 243. **4.** 336. **5.** 462. **6.** 286.

7. 55, 165. **8.** $\lfloor\underline{15} \div (\lfloor\underline{3})^5$. **9.** 30. **10.** 720, 60480.

11. 3168. **12.** $\lfloor\underline{20} \div \lfloor\underline{10} \lfloor\underline{10}$. **13.** $n(n^r - 1) \div (n - 1)$.

14. $\frac{1}{2}n(n-1) - \frac{1}{2}p(p-1) + 1$; $\frac{1}{6}n(n-1)(n-2) - \frac{1}{6}p(p-1)(p-2)$.

15. 6720. **16.** 6. **18.** 33. **19.** 83. **20.** 6, 10080.

22. 106560. **24.** 27720. **25.** 39916800.

26. 420. **28.** 80. 7. **29.** 2^n.

Exercise 37 (page 150)

1. $x^2 + x(a + b) + ab$. **2.** $x^3 + 6x^2 + 11x + 6$.

3. $x^4 + x^3(a + b + c + d) + x^2(ab + ac + ad + bc + bd + cd) + x(abc + bcd + cda + dab) + abcd$.

4. $a^3 + 3a^2b + 3ab^2 + b^3$.

5. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, $x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5$.

6. $a^{10} + 10a^9x + 45a^8x^2$, $x^{15} + 15x^{14}y + 105x^{13}y^2$.

7. $300a^{23}b^2$, $105m^{13}n^2$. **8.** x^3, 4th, 35. **9.** $792x^5a^7$.

10. 5, 8, 21.

Exercise 38 (page 153)

1. $a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6$.

2. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

3. $a^6 + 12a^5x + 60a^4x^2 + 160a^3x^3 + 240a^2x^4 + 192ax^5 + 64x^6$.

4. $1 + 4x^2 + 6x^4 + 4x^6 + x^8$.

5. $16x^4 - 32x^3a^2 + 24x^2a^4 - 8xa^6 + a^8$.

6. $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$.

7. $x^4 - 2x^3 + \frac{3}{2}x^2 - \frac{1}{2}x + \frac{1}{16}$.

8. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

Exercise 33 (page 153) *continued*

9. $x^{12} + 12x^{11}a + 66x^{10}a^2 + 220x^9a^3$, $a^8 - 16a^7b + 112a^6b^2$.

10. $2x^4 + 12x^2a^2 + 2a^4$, $10x + 20x^3 + 2x^5$.

11. 34, $2\sqrt{b}(5a^2 + 10ab + b^2)$.

12. $35a^4x^3$, $1215a^4b^2$, $-144x^2y^7$, $-96096x^{15}$.

13. $924x^{18}$, 70, $-4320x^3y^3$. **14.** $10a^3x^2$, $10a^2x^3$; $252m$, $-\frac{63}{m}$.

15. $252x^5a^5$, $2^8 \cdot x^8 \cdot \lfloor\underline{20} \div \lfloor\underline{8} \lfloor\underline{12}$, $-1792a^3b^5$.

16. $1 + 5x + 7x^2 - 5x^3 - 22x^4$.

17. $1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$.

18. $1 - 15x + 95x^2 - 330x^3$, $1 - 14x + 83x^2$.

19. 121. **20.** 48. **21.** -68.

Exercise 39 (page 155)

1. $\dfrac{\lfloor\underline{15}}{\lfloor\underline{r} \lfloor\underline{15-r}} x^{3r}$. **2.** $\dfrac{\lfloor\underline{n+1}}{\lfloor\underline{r} \lfloor\underline{n-r+1}} x^r$. **3.** $\dfrac{\lfloor\underline{m}}{\lfloor\underline{r} \lfloor\underline{m-r}} a^{m+r}$.

4. $\dfrac{\lfloor\underline{n}}{\lfloor\underline{r} \lfloor\underline{n-r}} (-1)^r x^{n-2r}$. **5.** 8008. **6.** $\lfloor\underline{2n} \div \lfloor\underline{n} \lfloor\underline{n}$.

7. -455. **8.** 66, 495, 220. **9.** The first and fourth.

11. $(-1)^{\frac{2n-p}{4}} \lfloor\underline{n} \div \lfloor\underline{\frac{2n-p}{4}} \lfloor\underline{\frac{2n+p}{4}}$, if $2n - p$ is a multiple of 4.

14. $a^{r-1}b^{n-r+1} \lfloor\underline{n} \div \lfloor\underline{r-1} \lfloor\underline{n-r+1}$. **15.** 9th.

16. $p + q = n + 2$. **17.** 6 **18.** $r = \frac{1}{3}(n + 1)$. **20.** 1, 3, 5.

21. ·9988. **22.** 1·0283.

Exercise 40 (page 159)

1. 16, 1024, 729, 5^n, 243. **2.** 64. **3.** 2^{n-1}, 2^{2n}.

Exercise 41 (page 162)

1. 26th. **2.** 12th and 13th. **3.** 1st. **4.** 7th.

5. 5th and 6th. **6.** $5\frac{1}{16}$, 8^7. **7.** 13th, 6th.

9. 9th and 10th are greater, all others are less.

Exercise 42 (page 162)

1. $(-1)^n \,|\underline{2n} \div |\underline{n}\,|\underline{n}$. **2.** $-a^7b^{13}\,|\underline{20} \div |\underline{13}\,|\underline{7}$.

3. $32 - 40x^2 + 10x^4$. **4.** $\frac{3^5}{2}a^{2^2} \div b^9$. **6.** 17th.

7. 220, 135. **8.** 4th, 11th. **10.** 145. **11.** 12. **13.** 7.

16. 20. **19.** 6th. **23.** 141. **25.** $|\underline{m+n} \div |\underline{r}\,|\underline{m+n-r}$.

Exercise 43 (page 168)

1. $1 + \frac{5}{2}x + \frac{15}{8}x^2 + \frac{5}{16}x^3$. **2.** $1 - 3x + 6x^2 - 10x^3$.

3. $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$. **4.** $1 - 2x^2 + 3x^4 - 4x^6$.

5. $1 + 4x + 10x^2 + 20x^3$. **6.** $\frac{1}{16}(1 + 2x + \frac{5}{2}x^2 + \frac{5}{2}x^3)$.

7. $81 - 12x + \frac{2}{3}x^2 + \frac{4}{729}x^3$.

8. $\frac{1}{x^5}(1 - 5x + 15x^2 - 35x^3)$. **9.** $1 - 2x - 2x^2 - 4x^3$.

10. $1 + x - x^2 + \frac{5}{3}x^3$. **11.** $1 - \frac{2}{3}x - \frac{1}{9}x^2 - \frac{4}{81}x^3$.

12. $a\left\{1 - \frac{x}{a^n} - \frac{n-1}{2} \cdot \frac{x^2}{a^{2n}} - \frac{(n-1)(2n-1)}{6} \cdot \frac{x^3}{a^{3n}}\right\}$.

13. $\frac{5}{2}x^3$. **14.** $-\frac{7}{48}x^4$. **15.** $21x^5$.

16. $-\frac{5}{1024}x^6$. **17.** $\frac{7}{1024}x^9$. **18.** $231x^6$.

20. $\frac{2}{5}$. **21.** 6th, 11th. **22.** 1st, 3rd. **23.** 2nd, 4th.

24. (*a*) Alternately plus and minus, (*b*) all plus, (*c*) all plus up to the eighth term, then alternately minus and plus, (*d*) alternately plus and minus up to the eighth term, thereafter all minus, (*e*) alternately plus and minus up to the ninth term, thereafter all plus.

Exercise 44 (page 170)

1. x^r. **2.** $\frac{1}{2}(r+1)(r+2)(-1)^r$. **3.** $\frac{2 \cdot 5 \cdot 8 \ldots (3r-1)}{\lfloor r}x^r$.

4. $(r+1)(-bx)^r a^{-r-2}$. **5.** $-\frac{1 \cdot 1 \cdot 3 \cdot 5 \ldots (2r-3)}{\lfloor r \; 2^r}x^r$.

6. $\frac{3 \cdot 1 \cdot 1 \cdot 3 \cdot 5 \ldots (2r-5)}{\lfloor r \quad 2^{3r-3}}x^r(-1)^r$. **7.** $-\frac{1 \cdot 2 \cdot 5 \cdot 8 \ldots (3r-4)}{\lfloor r}x^r$.

8. $\frac{1 \cdot 3 \cdot 5 \ldots (2r-1)}{\lfloor r}2^r \cdot x^r(-1)^r$.

9. $\frac{1(n-1)(2n-1) \ldots (nr-n-1)}{\lfloor r \quad a^{nr-1}}x^r(-1)^{r-1}$

14. $\frac{1}{6}(r+1)(r+2)(r+3)$. **16.** $\lfloor 2n \div \lfloor r \lfloor 2n-r$.

Exercise 45 (page 174)

1. 2nd. **2.** 6th, 7th. **3.** 2nd. **4.** 1st. **5.** 4th, 5th.

6. 99th, 100th.

8. $1 - 3x + 9x^2 - 27x^3 + \ldots, \frac{1}{3x} - \frac{1}{9x^2} + \frac{1}{27x^3} - \frac{1}{81x^4} + \ldots$

9. $x^4 + 4x^3 + 10x^2 + 20x + \ldots, x^8 + 4x^9 + 10x^{10} + 20x^{11} + \ldots$

12. 2·8284. **19.** $\frac{1}{6}$. **20.** 16.

Exercise 46 (page 176)

1. 80, $4r$. **2.** 631, $\frac{1}{2}(3r^2 + 3r + 2)$.

3. 25, $r + 5$. **4.** $-37, (3-2r)(-1)^r$.

5. $1 - x + 2x^2$, $1 - x - x^2$, $2 + x - \frac{3}{2}x^2$. **6.** $1\frac{3}{8}$.

7. 5252, $\frac{1}{2}(r+1)(r+4)(-1)^r$. **10.** $1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{25}{16}x^3$.

11. 5, 10. **12.** 1, -2, 1. **13.** 34. **14.** 1, -1, 0.

15. $\frac{1}{2}(n+1)(n+2)$. **17.** $\lfloor 2m \div \lfloor m \lfloor m$.

Exercise 47 (page 180)

1. 10·0166. **2.** ·98806. **3.** ·0080096. **4.** 2·0385.

7. 4·062. **9.** \$1·56. **10.** 996007. **11.** 9876.

12. ·0000576. **13.** ·0000189. **14.** 1·00028 sq. in. **15.** $1 + 7x$.

16. 0. **17.** $1 + \frac{1}{2}x$. **18.** $1 + \frac{5}{24}x$. **19.** $1 + 16x$.

20. $1 + 15x^2$. **21.** 165, $\lfloor \underline{n + r - 1} \div \lfloor \underline{n} \rfloor \underline{r - 1}$.

22. 8855. **23.** 1820. **24.** 10, 60.

Exercise 48 (page 181)

2. 4·996. **3.** $9r + 1$. **4.** $\frac{1}{6}(n + 1)(n + 2)(2n + 3)$.

5. 1·41421. **7.** $1 + \frac{17}{6}x - \frac{19}{30}x^2$. **8.** $39\frac{2}{7}\frac{4}{2}\frac{1}{5}$. **11.** $\frac{1}{2}\sqrt{5}$.

12. $\lfloor \underline{25} \div \lfloor \underline{5} \rfloor \underline{7} \rfloor \underline{13}$. **14.** ·000051. **17.** $2 + \frac{3}{4}\frac{c^2}{l^2}$.

18. $2^n + 2$. **20.** $\frac{x^n}{1 - x}$, $\frac{x^n}{(1 - x)^2} + \frac{nx^n}{1 - x}$.

27. $2^{n-3}(n^2 + 7n + 8)$. **28.** $2^{n-1}(n + 2r)$.

29. $3 \cdot 5 \cdot 7 \ldots (2n - 1) \div 2^{n-1} \lfloor \underline{n - 1}$. **30.** 5050.

NOTE.—The answers to the examples in the next three exercises were obtained by using seven-figure tables. If five-figure tables are used the result cannot usually be depended upon to more than four significant digits.

Exercise 49 (page 186)

1. \$125(1·03)^7, \$200(1·055)^4, \$720(1·02)^9, \$800(1·03)^{12}(1·015).

2. \$1000 ÷ (1·05)^5, \$360 ÷ (1·06)^4(1·03), \$820 ÷ (1·0225)^{11}.

3. \$213·83. **4.** \$123·33. **5.** \$179·08, \$180·61, \$181·40.

6. \$285·43. **7.** 14·2 years, 23·5 years. **8.** 21·2 years.

9. 20·2 years. **10.** 8·243%. **12.** \$1024. **13.** 19 years.

14. \$846·92.

Exercise 50 (page 191)

1. \$2387·57. **2.** \$1248·63. **3.** \$2330·94. **4.** \$5000.
5. \$1822·96. **6.** \$7860·18. **7.** \$3700·88. **8.** \$1960·66.
9. \$3290·81. **10.** \$4449·03. **11.** \$1000, \$1025·27.
12. \$1127·41. **13.** \$497·75. **14.** \$4488·22. **15.** \$4200.

Exercise 51 (page 195)

1. \$4203·03. **2.** \$2675·49. **3.** \$92·64. **4.** \$91·65.
5. \$120. **6.** \$962·91. **7.** \$839·54. **8.** \$942·21, \$962·82.
9. \$2513·20. **10.** \$19·92. **11.** \$77·88. **12.** \$8063·72.
13. \$230·48. **14.** \$105·23, \$120·06, \$99·88, \$233·96.
16. ·7265 of a mill.

Exercise 52 (page 198)

1. 5. **2.** 7·8. **3.** $P\{1-(1+i)^{-n}\}$. **4.** 19.
5. $\$5000(1-1{\cdot}03^{-n})$. **6.** $\frac{P}{i+\frac{1}{4}i^2}\{1-(1+\frac{1}{2}i)^{-2n}\}$.
7. $\frac{A^2}{A-B}\left\{\left(\frac{A}{B}\right)^{\frac{1}{n}}-1\right\}$. **8.** \$97·56. **9.** 28 years, nearly.
10. 19. **11.** $P = A \div (1{\cdot}03^{20}+1{\cdot}03^{10})$. **12.** 9, \$508·68.
14. \$1200, 4%. **15.** $P\left\{\left(1+\frac{r}{100m}\right)^m-\left(1+\frac{r}{100}\right)\right\}$.
16. $\frac{Pa}{b-a}$. **17.** \$1000, \$995·03.
18. $\frac{1}{2i+\frac{1}{2}i^2}\{1-(1+\frac{1}{2}i)^{-2n}\}$, $\frac{1}{2i+\frac{1}{2}i^2}$, n. **19.** \$464·97.
20. \$2091·51. **21.** Over 29 years.

Exercise 53 (page 202)

1. $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$. **2.** $\frac{1}{6}$, $\frac{4}{6}$. **3.** $\frac{1}{4}$. **4.** ·57917, ·42083.
5. $\frac{7}{19}$, $\frac{3}{19}$, $\frac{9}{19}$. **6.** $\frac{1}{66}$. **7.** $\frac{1}{120}$, $\frac{119}{120}$, $\frac{7}{24}$. **8.** $\frac{2}{5}$. **10.** $\frac{1}{3}$. **12.** $\frac{5}{42}$.

Exercise 54 (page 205)

1. $\frac{1}{15}$. **2.** $\frac{1}{4}$. **3.** $\frac{6}{35}$. **4.** $\frac{1}{6}$. **5.** $\frac{23}{60}$, $\frac{37}{60}$. **6.** $\frac{2}{15}$, $\frac{2}{5}$, $\frac{4}{15}$, $\frac{3}{5}$.

7. $\frac{6}{11}$. **8.** $\frac{2}{45}$. **9.** $p(1-q)$, pq, $1-pq$.

Exercise 55 (page 208)

1. \$18·75. **2.** \$800. **3.** \$6000. **4.** ·90652. **5.** ·8297.

6. ·6003. **7.** \$922·95. **8.** \$3007. **9.** ·81012. **10.** \$12·13.

11. \$24·40. **12.** ·6923. **13.** ·8153. **14.** \$36·00.

Exercise 56 (page 212)

1. 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23.
1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33.
1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.

2. 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44; 10, 11, 12, 20, 21, 22.

3. 12001. **4.** 5014. **5.** 23397. **6.** 5306421.

7. 6035044, 365031, 23001. **8.** 135. **10.** 325. **11.** 45, 123, 534.

12. 101010111. **13.** 3483. **14.** 13. **15.** 1236544.

Exercise 57 (page 214)

1. 202. **2.** 3231. **3.** 787. **4.** 17836. **5.** 69149.

6. 1000111101. **7.** 402854. **8.** 624.

9. $2 + 5 \cdot 9 + 7 \cdot 9^2 + 3 \cdot 9^3 + 6 \cdot 9^4$. **10.** 89197. **11.** 2342.

13. 14640, 1331. **14.** 165. **15.** 114381. **17.** 503.

Exercise 58 (page 216)

1. ·13, ·2, ·3. **3.** ·31, ·4513, ·64, ·99. **4.** ·754.

5. ·625, ·6, ·75. **6.** ·16, ·11513. **7.** 77·77. **8.** 969·375.

9. 6, 12. **10.** 1024. **11.** $\cdot\dot{3}$. **12.** ·2456.

Exercise 59 (page 220)

1. 5, 8, 0. **2.** 4. **3.** (2), (3). **7.** 8. **8.** 4.
9. 8, 5. **12.** 13. **13.** 17. **14.** 6. **15.** 12.
16. 9. **18.** 2^3, 2^5, 2^6, 2^7, 2^8, 2^9. **19.** $\frac{5}{7}$. **21.** 146 sq. ft. 82 sq. in.
23. 132. **26.** 83·2 cu. ft.

Miscellaneous Examples (page 222)

3. (2, 3, 2), $(1\frac{1}{2}, 1, \frac{1}{2})$. **4.** (1, 1), $(\frac{8}{15}, \frac{1}{15})$.
6. (1, 2, 3), $(\frac{3}{10}, \frac{5}{6}, \frac{2}{3})$. **7.** $x^3 + x^2 + 2x - 3$.
9. $\frac{3}{5}$. **10.** (4, 1). **11.** $(a - b)(b - c)(c - a)(a + b + c)$.
15. 55. **16.** 2, -4, $-1 \pm \sqrt{71}$. **18.** $b = a^2 + d$, $c = ad$.
19. $n^2(b - a)$. **22.** 10, 24, 26 in. **24.** 6120. **25.** 12.
26. $\frac{2}{3}$. **27.** $1\frac{1}{20}n(n - 1)(n + 1)(n + 2)(n + 3)$. **32.** $10\frac{1}{2}$.
33. 6. **34.** 5, -1, -3. **35.** nx^{n-1}.
37. ± 4, ± 8, ∓ 6. **38.** $(ca_1 - c_1a)^3 = (ab_1 - a_1b)(bc_1 - b_1c)^2$.
41. $a^3 + 2c^3 = 3ab^2$. **42.** a, b, $\frac{1}{2}(a + b)$. **45.** 1, 2, 3 ft.
46. (1, 2, 3), $(1\frac{7}{5}, \frac{88}{25}, -\frac{9}{25})$. **47.** 1·68, $-$ ·82.
49. 50, 8, 18. **50.** $a^6 + 9a^2c^4 = 2b^c + 8a^3b^3$.
51. $3(x - 3)^3 + 34(x - 3)^2 + 108(x - 3) + 68$.
53. $-pq(2p + 2q - 1)$. **54.** \$899·76.
56. $(a^2 + bc)(b^2 + ca)(c^2 + ab)$.
57. (1, 2, 4), (1, 4, 2), etc. **58.** $\frac{10}{81}(10^n - 1) - \frac{1}{9}n$.
59. $\frac{1}{2}(1 \pm \sqrt{5})$. **60.** $\left(a^2 \cdot \frac{a^n - 1}{a - 1} - b^2 \cdot \frac{b^n - 1}{b - 1}\right) \div (a - b)$.
61. 70. **62.** $p^2(b^2 - ac) = a^2(q^2 - pr)$. **64.** $-3\frac{1}{2}$.
65. $\frac{1}{12}n(n + 4)(2n + 1)$, $\frac{1}{12}(n + 1)(2n^2 + n + 3)$.
67. $\sqrt{mb} \pm \sqrt{nc} \pm \sqrt{al} = 0$. **69.** 76. **70.** $1\frac{1}{6}$.
71. 1, 4, 1, 0. **75.** (2, 3, 1), $(3, 2\frac{1}{2}, -1)$. **77.** 2^n.
78. 31. **81.** 2, $2\frac{1}{2}$, 3. **83.** $3^r - 2^r$. **87.** 56.
88. 246, 792. **89.** $\frac{1}{12}n(n + 1)(n + 2)(n + 3)$. **93.** 7.
96. 1, 3, 5, . . . **101.** $y^2 = a^2 + x^2 + 6ax$.

Miscellaneous Examples (page 222) *continued*

102. 2. 104. $(q - q_1)^3 = (pq_1 - p_1q)(p_1 - p)^2$.

105. \$95·18. 106. 60. 107. $5 - \sqrt{2}$, $1\frac{1}{3}$, -3.

109. 22, 286. 110. $\frac{1}{3}\frac{1}{1}$. 111. $6n^2 - 22n + 9$.

112. $a - c - d$. 113. (6, 1), (− 1, − 6). 115. 110.

116. 11, 7. 117. 274. 119. $n(n + 3)2^{n-3}$.

123. 8, 0. 124. $an + d(2^n - n - 1)$. 129. 30 miles.

131. $\frac{n}{a} - \frac{bn(n + 1)}{2a^2}$. 132. $\frac{1}{2}mn(m + n)$.

134. $\frac{1}{3}(m + 1)(2m^2 + 4m + 3)$. 139. $\frac{1}{6}\frac{1}{0}(n + 1)(n + 2)\ldots(n + 6)$.

140. $\underline{|m + n - 1} \div \underline{|m}\,|\,n - 1$. 142. $\frac{7}{2}\frac{7}{5}p$, if $\pi = 3\frac{1}{7}$.

151. $1 - \frac{6}{(n + 1)(n + 2)(2n + 3)}$. 152. 3.

157. $2(b^3 - d^3) = 3(b^2 - c^2)(b - a)$.

158 $\frac{1}{6}\frac{1}{0}n(n + 1)(2n + 1)(3n^2 + 3n + 4)$. 161. 1, $\frac{2}{3}$, $\frac{3}{2}$, − 2, $-\frac{1}{2}$.

163. $2^{n-3}(n^2 + 7n + 8)$. 168. 792. 171. 19 years.

181. ${}_{p+q+r}C_3 - {}_{q+1}C_3 - {}_{p+1}C_3$.

The Hunter-Rose, Co., Limited, Toronto.

THE MACMILLAN COMPANY OF CANADA LTD